

LIMITING ABSORPTION PRINCIPLE FOR SCHRÖDINGER OPERATORS WITH OSCILLATING POTENTIAL

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ABSTRACT. Making use of the weighted Mourre theory developed in [GJ1], we show the limiting absorption principle for Schrödinger operators with perturbed oscillating potential on appropriate energy intervals. We focus on a certain class of oscillating potentials (larger than the one in [GJ2]) that was already studied in [BD, MU, ReT1, ReT2]. We allow long-range and short-range components and local singularities in the perturbation. We improve known results, the main novelty being the presence of a long-range perturbation. A subclass of the considered potentials actually cannot be treated by the usual Mourre commutator method. Inspired by [FH], we also show, in some cases, the absence of positive eigenvalues for our Schrödinger operators.

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1. INTRODUCTION.

In this paper, we are interested in the behaviour near the positive real axis of the resolvent of a class of continuous Schrödinger operators. We shall prove a so called “limiting absorption principle”, a very useful result to develop the scattering theory associated to those Schrödinger operators. It also gives informations on the nature of their essential spectrum, as a byproduct. The main interest of our study relies on the fact that we include some oscillating contribution in the potential of our Schrödinger operators.

To set up our framework and precisely formulate our results, we need to introduce some notation. Let $d \in \mathbb{N}^*$. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the right linear scalar product and the norm in $L^2(\mathbb{R}^d)$, the space of squared integrable, complex functions on \mathbb{R}^d . We also denote by $\| \cdot \|$ the norm of bounded operators on $L^2(\mathbb{R}^d)$. Writing $x = (x_1; \dots; x_d)$ the variable in \mathbb{R}^d , we set

$$\langle x \rangle := \left(\sum_{j=1}^d x_j^2 \right)^{1/2}.$$

Let Q_j the multiplication operator in $L^2(\mathbb{R}^d)$ by x_j and P_j the self-adjoint realization of $-i\partial_{x_j}$ in $L^2(\mathbb{R}^d)$. We set $Q = (Q_1; \dots; Q_d)^T$ and $P = (P_1; \dots; P_d)^T$, where T denotes the transposition. Let

$$H_0 = |P|^2 := \sum_{j=1}^d P_j^2 = P^T \cdot P$$

be the self-adjoint realization of the nonnegative Laplace operator $-\Delta$ in $L^2(\mathbb{R}^d)$. We consider the Schrödinger operator $H = H_0 + V(Q)$, where $V(Q)$ is the multiplication operator by a real valued function V on \mathbb{R}^d satisfying the following

Assumption 1.1. *Let $\alpha, \beta \in]0; +\infty[$. Let $\rho_{sr}, \rho_{lr}, \rho'_{lr} \in]0; 1]$. Let $V_{sr}, V_{lr}, V_c, W_{\alpha\beta} : \mathbb{R}^d \rightarrow \mathbb{R}$ such that V_c is compactly supported, $V_c(Q)$ is H_0 -compact, the functions $\langle x \rangle^{1+\rho_{sr}} V_{sr}(x)$, $\langle x \rangle^{\rho_{lr}} V_{lr}(x)$ and the distribution $\langle x \rangle^{\rho'_{lr}} x \cdot \nabla V_{lr}(x)$ are bounded, and*

$$(1.1) \quad W_{\alpha\beta}(x) = w(1 - \kappa(|x|))|x|^{-\beta} \sin(k|x|^\alpha)$$

with real w and a function $\kappa \in C_c^\infty(\mathbb{R}; \mathbb{R})$ such that $\kappa = 1$ on $[-1; 1]$ and $0 \leq \kappa \leq 1$. Let $V = V_{sr} + V_{lr} + V_c + W_{\alpha\beta}$.

Under Assumption 1.1, $V(Q)$ is H_0 -compact. Therefore H is self-adjoint on the domain $\mathcal{D}(H_0)$ of H_0 , which is the Sobolev space of $L^2(\mathbb{R}^d)$ -functions such that their distributional derivative up to second order belong to $L^2(\mathbb{R}^d)$. By Weyl's theorem, the essential spectrum of H is given by the spectrum of H_0 , namely $[0; +\infty[$. To formulate our first main result, we shall need the following

Assumption 1.2. *Let $\alpha \geq 1 \geq \beta$. Setting $\beta_{lr} = \min(\beta; \rho_{lr})$, we take β and ρ_{lr} such that $\beta + \beta_{lr} > 1$ or, equivalently, $\beta > 1/2$ and $\rho_{lr} > 1 - \beta$. We consider a compact interval \mathcal{I} such that $\mathcal{I} \subset]0; k^2/4[$, if $\alpha = 1$, and $\mathcal{I} \subset]0; +\infty[$, if $\alpha > 1$.*

Remark 1.3. If $\beta > 1$, $W_{\alpha\beta}$ can be considered as short range potential like V_{sr} . If $\alpha < \beta \leq 1$, $W_{\alpha\beta}$ satisfies the long range condition required on V_{lr} . Our assumptions allow V to contain the function $x \mapsto |x|^{-\beta} \sin(k|x|^\alpha)$ with $\beta < 2 + \alpha$. This function was considered in [BD, DMR, DR1, DR2, ReT1, ReT2]. Assumption 1.2

excludes the situation where $0 < \beta \leq \alpha < 1$. A reason for this is given just after Proposition 2.1 in Section 2. It turns out that our results do not change if one replaces the sinus function in $W_{\alpha\beta}$ by a cosinus function.

Remark 1.4. In [Co, CG], a certain class of potentials that can be written as the divergence of a short range potential were studied. Those potentials actually can be handled by the usual Mourre theory (they have a good enough regularity w.r.t. the generator of dilations). For $\alpha = \beta = 1$, it was proved in [GJ2] that H does not match the required assumptions of the usual Mourre theory. In particular, W_{11} cannot be written as the divergence of a short range potential.

Let Π be the orthogonal projection onto the pure point spectral subspace of H . We set $\Pi^\perp = 1 - \Pi$. For any complex number $z \in \mathbb{C}$, we denote by $\Re z$ (resp. $\Im z$) its real (resp. imaginary) part. Our first main result is the following limiting absorption principle (LAP).

Theorem 1.5. *Suppose Assumptions 1.1 and 1.2 are satisfied. For any $s > 1/2$,*

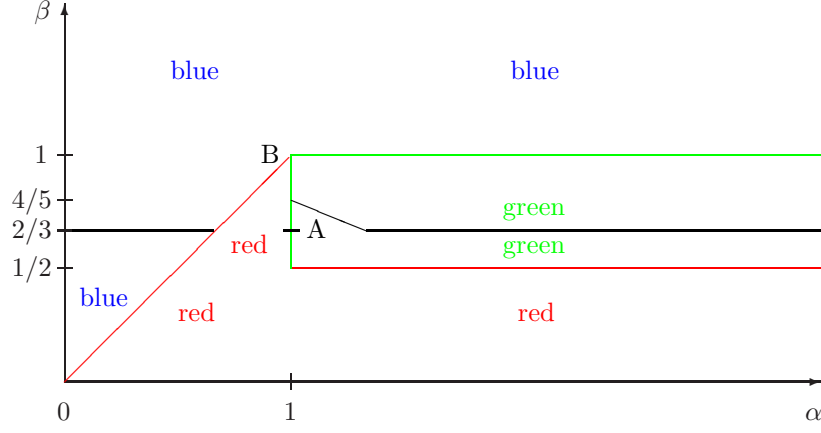
$$(1.2) \quad \sup_{\substack{\Re z \in \mathcal{I}, \\ \Im z \neq 0}} \|\langle Q \rangle^{-s} (H - z)^{-1} \Pi^\perp \langle Q \rangle^{-s}\| < +\infty.$$

Remark 1.6. In the literature, the LAP is often proved away from the point spectrum. If \mathcal{I} in (1.2) does not intersect the latter, one can remove Π^\perp in (1.2) and therefore get the usual LAP. But the LAP (1.2) gives an information on the absolutely continuous subspace of H near possible embedded eigenvalues.

Historically, LAPs for Schrödinger operators were first obtained by perturbation, starting from the LAP for the Laplacian H_0 . Lavine initiated nonnegative commutator methods in [La1, La2] by adapting Putnam's idea (see [CFKS] p. 60). Mourre introduced 1980 in [Mo] a powerful, non perturbative, local commutator method, nowadays called “Mourre commutator theory” (see [ABG, GGé, GGM, JMP, Sa]). Nevertheless it cannot be applied to potentials that contain some kind of oscillatory term (cf. [GJ2]). In [Co, CG], the LAP was proved perturbatively for a class of oscillatory potentials that differs from the one considered here, as pointed out in Remark 1.4. In [BD, DMR, DR1, DR2, ReT1, ReT2], the present situation with $V_c = 0$ and a radial long range contribution V_{lr} was treated using tools of ordinary differential equations and again a perturbative argument. Theorem 1.5 improves the results of these papers in two ways. First, we allow a long range (non radial) part in the potential. Second, the set \mathcal{V} of values of $(\alpha; \beta)$, for which the LAP (on some interval) holds true, is here larger. However, in the case $\alpha = 1$, these old results provide a LAP also beyond $k^2/4$ in all dimension d , whereas we are able to do so only in dimension $d = 1$. For $\alpha = \beta = 1$, the LAP at high enough energy was proved in [MU]. Another proof of this result is sketched in Remark 1.8 below.

We point out that the discrete version of the present situation is treated in [Man]. We also signal that the LAP for continuous Schrödinger operators is studied in [Mar] by Mourre commutator theory but with new conjugate operators, including the one used in [N].

In Fig. 1, we drew the set \mathcal{V} in a $(\alpha; \beta)$ -plane. The papers [BD, DMR, DR1, DR2, ReT1, ReT2] established the LAP in the region above the red and black lines and, along the vertical green line, above the point $A = (1; 2/3)$. The usual Mourre

FIGURE 1. LAP. $\mathcal{V} = \text{blue} \cup \text{green}$.

theory proves it in the blue region (above the red lines and the green one). Both results are obtained without energy restriction. Theorem 1.5 covers the blue and green regions (the set \mathcal{V}), with a energy restriction on the vertical green line. In [GJ2], the LAP with energy restriction is proved at the point $B = (1; 1)$. In the red region (below the red lines), the LAP is still an open question.

Let A be the self-adjoint realization of the operator $(P \cdot Q + Q \cdot P)/2$ in $L^2(\mathbb{R}^d)$. We are able to get the following improvement of a main result in [GJ2].

Theorem 1.7. *Let $\alpha = \beta = 1$. Under Assumption 1.1 with $V_c = 0$, take a compact interval $\mathcal{I} \subset]0; k^2/4[$. Then, for any $s > 1/2$,*

$$(1.3) \quad \sup_{\substack{\Re z \in \mathcal{I}, \\ \Im z \neq 0}} \|\langle A \rangle^{-s} (H - z)^{-1} \Pi^\perp \langle A \rangle^{-s}\| < +\infty.$$

Proof. In [GJ2], it was further assumed that, for any $\mu \in \mathcal{I}$, $\text{Ker}(H - \mu) \subset \mathcal{D}(A)$. Thanks to Corollary 4.2, this assumption is superfluous. \square

Remark 1.8. Note that Assumption 1.2 is satisfied for $\alpha = \beta = 1$. In dimension $d = 1$, the above result is still true if $\mathcal{I} \subset]k^2/4; +\infty[$. A careful inspection of the proof in [GJ2] shows that Theorem 1.7 holds true in all dimensions if $\mathcal{I} \subset]a; +\infty[$, for large enough positive a (depending on $|w|$). If $|w|$ is small enough, the mentioned proof is even valid on any compact interval $\mathcal{I} \subset]0; +\infty[$.

For nonzero potential V_c , we believe that one can adapt the proof in [GJ2] of Theorem 1.7.

Remark 1.9. It is well known that (1.3) implies (1.2). Let us sketch this briefly. It suffices to restrict s to $]1/2; 1[$. Take $\theta \in C_c^\infty(\mathbb{R}; \mathbb{R})$ such that $\theta = 1$ near \mathcal{I} . Then, the bound (1.2) is valid if $(H - z)^{-1}$ is replaced by $(1 - \theta(H))(H - z)^{-1}$. The boundedness of the contribution of $\theta(H)(H - z)^{-1}$ to the l.h.s of (1.2) follows from (1.3) and from the boundedness of $\langle Q \rangle^{-s} \theta(H) \langle A \rangle^s$. To see the last property, one can write

$$\langle Q \rangle^{-s} \theta(H) \langle A \rangle^s = \langle Q \rangle^{-s} \theta(H) \langle P \rangle^s \langle Q \rangle^s \cdot \langle Q \rangle^{-s} \langle P \rangle^{-s} \langle A \rangle^s.$$

The last factor is bounded by Lemma C.1 in [GJ2]. The boundedness of the other one is granted by the regularity of H w.r.t. $\langle Q \rangle$ (see Section 3) and the fact that $\theta(H)\langle P \rangle$ is bounded.

Remark 1.10. It is well known that (1.2) implies the absence of singular continuous spectrum in \mathcal{I} (see [RS4]). On this subject, we refer to [K, Rem].

As already pointed out, one cannot use the usual Mourre commutator theory (cf. [Mo, ABG]) to prove Theorem 1.5 or Theorem 1.7. The given proof of Theorem 1.7 relies on a kind of “energy localised Putnam” argument. This method, which is reminiscent of the works [La1, La2] by Lavine, was introduced in [GJ1] and improved in [Gé, GJ2]. It was originally called “weighted Mourre theory” but it is closer to Putnam idea (see [CFKS] p. 60) and does not make use of differential inequalities as the original Mourre theory. However the latter gives stronger results than the former. We did not succeed in applying the “localised Putnam theory” formulated in [GJ2] to prove Theorem 1.5. We believe that the bad regularity of the operator w.r.t. the generator of dilations is the source of our difficulties. Instead, we follow the more complicated version presented in [GJ1], which relies on a Putnam type argument that is localized in Q and H .

As an intermediate step in the proof of Theorems 1.5 and 1.7, we shall prove that the pure point spectrum of H is locally finite (counting multiplicity) in $]0; +\infty[$, if $\alpha > 1$, and in $]0; k^2/4[$, if $\alpha = 1$ (cf. Corollary 5.2). Adapting arguments from [FH], we shall get polynomial bounds on the possible eigenvectors of H with energy in $]0; +\infty[$, if $\alpha > 1$, and in $]0; k^2/4[$, if $\alpha = 1$ (see Proposition 6.1). In the case $\alpha > 1$, we even show exponential bounds. Inspired by the paper [FHHH2], we shall derive our second main result, namely

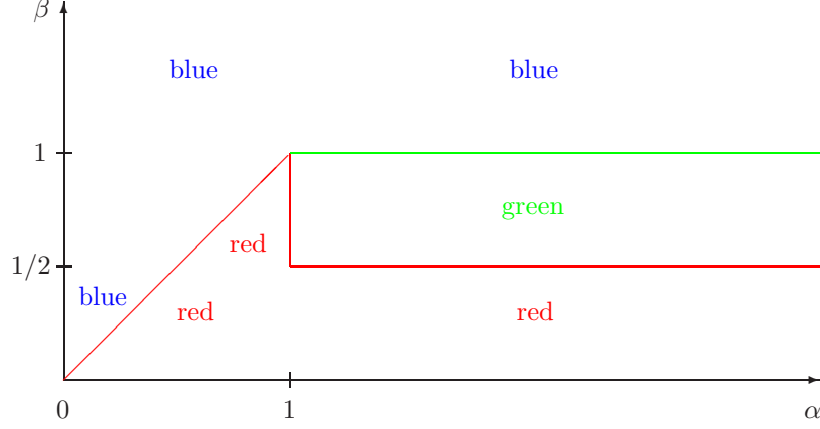
Theorem 1.11. *Under Assumptions 1.1 and 1.2 with $\alpha > 1$, we assume further that the form $[V_c, iA]$ is H_0 -form-lower-bounded with relative bound < 2 (see (7.1) for details). Then H has no positive eigenvalue.*

Proof. The result follows from Propositions 6.1 and 7.2. □

Remark 1.12. When $\beta > 1$, the absence of positive eigenvalue follows from [FH]. But neither the results in [FH] nor the ones in [FHHH2] apply in the present situation. However we essentially follow the strategies used in these papers to perform our proofs. A similar assumption to (7.1) appears in [FHHH2]. The main new ingredient in the proof is an appropriate control on the oscillatory part of the potential.

Remark 1.13. In the case $\alpha = \beta = 1$, assuming (7.1), we can show the absence of eigenvalue at high energy. This follows from Remark 6.3 and Proposition 7.2. However an embedded eigenvalue does exist for an appropriate choice of V (see [FH, CFKS, CHM]).

Remark 1.14. Under the assumptions of Theorem 1.11, for any compact interval $\mathcal{I} \subset]0; +\infty[$, the result of Theorem 1.5, namely (1.2), is valid with Π^\perp replaced by the identity operator. Indeed, for any compact interval $\mathcal{I}' \subset]0; +\infty[$ containing \mathcal{I} in its interior, $\mathbb{1}_{\mathcal{I}'}(H)\Pi = 0$ by Theorem 1.11. In view of Remark 1.8, the LAP (1.3) is valid at high energy, when $\alpha = \beta = 1$. Thanks to Remark 1.13, one can also remove Π^\perp in (1.3).

FIGURE 2. No positive eigenvalue in $\text{blue} \cup \text{green}$.

One can find many papers on the absence of positive eigenvalue for Schrödinger operators: see for instance [Co, K, Si, A, FHHH2, FH, IJ, RS4, CFKS]. They do not cover the present situation due to the oscillations in the potential. In Fig. 2, we summarise results on the absence of positive eigenvalue. In the blue region (above the red and green lines), the result follows from [FHHH2, FH]. Theorem 1.11 covers the blue and green region (above the red lines).

In Assumption 1.2, the parameter ρ_{lr} , that controls the behaviour at infinity of the long range potential V_{lr} , stays in a β -dependent region. One can get rid of this constraint if one chooses a smooth, symbol-like function as V_{lr} , as seen in the next

Theorem 1.15. *Assume that Assumption 1.1 is satisfied with $\beta > 1/2$ and with the further requirement that $V_{lr} : \mathbb{R}^d \rightarrow \mathbb{R}$ is a smooth function such that, for some $\rho_{lr} \in]0; 1]$, for all $\gamma \in \mathbb{N}^d$,*

$$\sup_{x \in \mathbb{R}^d} |\langle x \rangle^{\rho_{lr} + |\gamma|} (\partial_x^\gamma V_{lr})(x)| < +\infty.$$

Take $\alpha = 1$. Then the LAP (1.2) holds true on any compact interval \mathcal{I} such that $\mathcal{I} \subset]0; k^2/4[$, if $d \geq 2$, and such that $\mathcal{I} \subset]0; +\infty[\setminus \{k^2/4\}$, if $d = 1$.

Take $\alpha > 1$. Then the LAP (1.2) holds true on any compact interval $\mathcal{I} \subset]0; +\infty[$. If, in addition, $[V_c, iA]$ is H_0 -form-lower-bounded with relative bound < 2 (see (7.1) for details), then H has no positive eigenvalue. In particular, (1.2) holds true with Π^\perp removed.

Remark 1.16. We expect that our results hold true for a larger class of oscillatory potential provided that the “interference” phenomenon exhibited in Section 2 is preserved. In particular, we do not need that $W_{\alpha\beta}$ is radial.

We point out that there still are interesting, open questions on the Schrödinger operators studied here. Concerning the LAP, for $\alpha = 1$, it is expected that (1.2) is false near $k^2/4$. Note that the Mourre estimate is false there, when $\beta = 1$ (see [GJ2]). The validity of (1.2) beyond $k^2/4$ is still open, even at high energy when

$\beta < 1$. Concerning the existence of positive eigenvalue, again for $\alpha = 1$, it is known in dimension $d = 1$ that there is at most one at $k^2/4$ if $\beta = 1$ (see [FH]). It is natural to expect that this is still true for $d \geq 2$ and $\beta = 1$. We do not know what happens for $\alpha = 1 > \beta$.

In Section 2, we analyse the interaction between the oscillations in the potential $W_{\alpha\beta}$ and the kinetic energy operator H_0 . In Section 3, in some appropriate energy window, we show the Mourre estimate, which is still a crucial result. We deduce from it polynomial bounds on possible eigenvectors of H in Section 4. This furnishes the material for the proof of Theorem 1.7. In Section 5, we show the local finiteness of the point spectrum in the mentioned energy window. In the case $\alpha > 1$, we show exponential bounds on possible eigenvectors in Section 6 and prove the absence of positive eigenvalue in Section 7. Independently of Sections 6 and 7, we prove Theorem 1.5 in Section 8. Section 9 is devoted to the proof of Theorem 1.15. Finally, we gathered well-known results on pseudodifferential calculus in Appendix A, basic facts on regularity w.r.t. an operator in Appendix B, known results on commutator expansions and technical results in Appendix C, and an elementary, but lengthy argument, used in Section 2, in Appendix D.

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2. OSCILLATIONS.

In this section, we study the oscillations appearing in the considered potential V . It is convenient to make use of some standard pseudodifferential calculus, that we recall in Appendix A. As in [GJ2], our results strongly rely on the interaction of the oscillations in the potential with localizations in momentum (i.e. in H_0). This interaction is described in the following two propositions.

The oscillating part of the potential V occurs in the potential $W_{\alpha\beta}$ as described in Assumption 1.1. By (1.1), for some function $\kappa \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{R})$ such that $\kappa = 1$ on $[-1; 1]$ and $0 \leq \kappa \leq 1$, $W_{\alpha\beta} = w(2i)^{-1}(e_+^\alpha - e_-^\alpha)$, where

$$(2.1) \quad e_\pm^\alpha : \mathbb{R}^d \longrightarrow \mathbb{C}, \quad e_\pm^\alpha(x) = (1 - \kappa(|x|))e^{\pm ik|x|^\alpha}.$$

Let g_0 be the metric defined in (A.2).

Proposition 2.1. [GJ2]. *Let $\alpha = 1$. For any function $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{C})$, there exist smooth symbols $a_\pm \in \mathcal{S}(1; g_0)$, $b_\pm, c_\pm \in \mathcal{S}(\langle x \rangle^{-1} \langle \xi \rangle^{-1}; g_0)$ such that*

$$(2.2) \quad e_\pm^\alpha \theta(H_0) = a_\pm^w e_\pm^\alpha + b_\pm^w e_\pm^\alpha + e_\pm^\alpha c_\pm^w$$

and, near the support of $1 - \kappa(|\cdot|)$, a_\pm is given by

$$a_\pm(x; \xi) = \theta\left(|\xi \mp \alpha k|x|^{\alpha-2}x|^2\right).$$

In particular, if θ has a small enough support in $]0; k^2/4[$, then, for any $\epsilon \in [0; 1[$, the operator $\theta(H_0)\langle Q \rangle^\epsilon \sin(k|Q|)\theta(H_0)$ extends to a compact operator on $L^2(\mathbb{R}^d)$, and it is bounded if $\epsilon = 1$.

Remark 2.2. In dimension $d = 1$, the last result in Proposition 2.1 still holds true if θ has small enough support in $]0; +\infty[\setminus \{k^2/4\}$ (see [GJ2]).

Proof of Proposition 2.1. See Lemma 4.3 and Proposition A.1 in [GJ2]. \square

In any dimension $d \geq 1$, for $0 < \alpha < 1$, the above phenomenon is absent. A careful inspection of the proof of (2.2) shows that it actually works if $0 < \alpha < 1$. But, in constrast to the case $\alpha = 1$, the principal symbol of $\theta(H_0)\langle Q \rangle^\epsilon \sin(k|Q|)\theta(H_0)$, which is given by

$$\mathbb{R}^{2d} \ni (x; \xi) \mapsto (2i)^{-1} \theta(|\xi|^2) (a_+ - a_-)(x; \xi),$$

is not everywhere vanishing, for any choice of nonzero θ with support in $]0; +\infty[$. The conditions “ $|\xi|^2$ in the support of θ ” and “ $|\xi \mp \alpha k|x|^{\alpha-2}x|^2$ in the support of θ ” are indeed compatible for large $|x|$.

In this setting, namely for $0 < \alpha < 1$ and $d \geq 1$, one can give the following, more precise picture with the help of an appropriate pseudodifferential calculus. Take a nonzero, smooth function θ with compact support in $]0; +\infty[$. For $\epsilon \in]0; 1[$, on $L^2(\mathbb{R}^d)$, the operator

$$\theta(H_0)\langle Q \rangle^\epsilon \sin(k|Q|^\alpha)\theta(H_0) \quad \left(\text{resp. } \theta(H_0) \sin(k|Q|^\alpha)\theta(H_0) \right)$$

is unbounded (resp. is not a compact operator). Indeed, for the function κ given in (2.1), the multiplication operator

$$(1 - \kappa(|Q|)) \sin(k|Q|^\alpha)$$

is a pseudodifferential operator with symbol in $\mathcal{S}(1; g_\alpha)$ for the metric g_α defined in (A.2). By pseudodifferential calculus for this admissible metric g_α , the symbol of

$$\theta(H_0)\langle Q \rangle^\epsilon (1 - \kappa(|Q|)) \sin(k|Q|^\alpha)\theta(H_0),$$

namely

$$\theta(|\xi|^2) \# \langle x \rangle^\epsilon (1 - \kappa(|x|)) \sin(k|x|^\alpha) \# \theta(|\xi|^2),$$

is not a bounded symbol. Thus, the operator is unbounded on $L^2(\mathbb{R}^d)$, while

$$\theta(H_0)\langle Q \rangle^\epsilon \kappa(|Q|) \sin(k|Q|^\alpha)\theta(H_0)$$

is compact since its symbol $\theta(|\xi|^2) \# \langle x \rangle^\epsilon \kappa(|x|) \sin(k|x|^\alpha) \# \theta(|\xi|^2)$ tends to 0 at infinity. Still for the metric g_α , the symbol of

$$\theta(H_0)(1 - \kappa(|Q|)) \sin(k|Q|^\alpha)\theta(H_0)$$

is $\theta(|\xi|^2) \# (1 - \kappa(|x|)) \sin(k|x|^\alpha) \# \theta(|\xi|^2)$, that does not tend to zero at infinity. Therefore $\theta(H_0)(1 - \kappa(|Q|)) \sin(k|Q|^\alpha)\theta(H_0)$ is not a compact operator, whereas so is $\theta(H_0)\kappa(|Q|) \sin(k|Q|^\alpha)\theta(H_0)$.

Remark 2.3. The difference between the cases $\alpha = 1$ and $0 < \alpha < 1$ sketched just above explains why we exclude the case $\beta \leq \alpha < 1$ in our results. Recall that the case $0 < \alpha < \beta \leq 1$ is covered by the usual Mourre theory (cf. Remark 1.3).

In the case $\alpha > 1$, one can relax the localization to get compactness as seen in

Proposition 2.4. *Let $\alpha > 1$. For any real $p \geq 0$, there exist $\ell_1 \geq 0$ and $\ell_2 \geq 0$ such that $\langle P \rangle^{-\ell_1} \langle Q \rangle^p (1 - \kappa(|Q|)) \sin(k|Q|^\alpha) \langle P \rangle^{-\ell_2}$ extends to a compact operator on $L^2(\mathbb{R}^d)$. In particular, so does $\theta(H_0)\langle Q \rangle^p (1 - \kappa(|Q|)) \sin(k|Q|^\alpha)\theta(H_0)$, for any p and any $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{C})$.*

Proof. The proof is rather elementary and postponed in Appendix D. Appropriate ℓ_1 and ℓ_2 depend on p , α , and on the dimension d . For instance, one can choose ℓ_1 and ℓ_2 greater than 1 plus the integer part of $(\alpha - 1)^{-1}(p + d)$. \square

Remark 2.5. Take $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{C})$, $\tau \in \mathcal{C}_c^\infty(\mathbb{R}^d; \mathbb{C})$ such that $\tau = 1$ near zero, and $\alpha > 2$. The smooth function

$$(x; \xi) \mapsto (1 - \tau(x))\theta\left(|\xi \mp \alpha k|x|^{\alpha-2}x|^2\right),$$

does not belong to $\mathcal{S}(m; g)$ for any weight m associated to the metric g_0 . So we cannot use the proof of Proposition 2.1 in this case.

The proof of Proposition 2.4 shows that the oscillations manage to transform a decay in $\langle P \rangle$ in one in $\langle Q \rangle$. This is not suprising if one is aware of the following, one dimensional formula (see eq. (VII. 5; 2), p. 245, in [Sc]), pointed out by V. Georgescu. For any $m \in \mathbb{N}$, there exist $\lambda_0, \dots, \lambda_{2m} \in \mathbb{C}$ such that

$$\forall x \in \mathbb{R}, \quad (1 + x^2)^m e^{i\pi x^2} = \sum_{j=0}^{2m} \lambda_j \frac{d^j}{dx^j} e^{i\pi x^2}.$$

Note that the result of Proposition 2.4 is false for $\alpha \leq 1$ by Proposition 2.1 and the discussion following it.

3. THE MOURRE ESTIMATE.

In this section, we establish a Mourre estimate for the operator H near appropriate positive energies. In the spirit of [FH], we deduce from it spacial decaying, polynomial bounds on the possible eigenvectors of H at that energies. Since H does not have a good regularity w.r.t. the conjugate operator A , the abstract setting of Mourre theory does not help much and we have to look more precisely at the structure of H . The properties derived in Section 2 play a key role in the result.

Recall that $A = (P \cdot Q + Q \cdot P)/2$. The regularity spaces $\mathcal{C}^k(A)$, for $k \in \mathbb{N}^* \cup \{\infty\}$, are defined in Appendix B. Since $[H_0, iA] = 2H_0$, as forms, one can easily show that $H_0 \in \mathcal{C}^\infty(A)$. The $\mathcal{C}^1(A)$ -regularity is natural to give a meaning to the Mourre estimate. But, if $\alpha > 1$, we expect that $H \notin \mathcal{C}^1(A)$ since the bounded potential $W_{\alpha\beta}$ does not belong to $\mathcal{C}^1(A)$ in this case (see below).

However, since $V(Q)$ commutes with $\langle Q \rangle$, we have at our disposal a much better regularity w.r.t. to $\langle Q \rangle$ as pointed out in

Lemma 3.1. *Assume that Assumptions 1.1 and 1.2 are satisfied.*

- (1) H_0 , $\langle P \rangle$, and P belong to $\mathcal{C}^\infty(\langle Q \rangle)$ and $\mathcal{D}(\langle Q \rangle \langle P \rangle) = \mathcal{D}(\langle P \rangle \langle Q \rangle)$.
- (2) $H \in \mathcal{C}^\infty(\langle Q \rangle)$.
- (3) For $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{C})$, $\theta(H_0)$, $P\theta(H_0)$, $\theta(H)$, and $P\theta(H)$ belong to $\mathcal{C}^\infty(\langle Q \rangle)$, and we have the inclusion $\theta(H)\mathcal{D}(\langle Q \rangle) \subset \mathcal{D}(\langle P \rangle \langle Q \rangle) \cap \mathcal{D}(H_0)$.

Proof. See Appendix C. \square

To study the formal commutator $[H, A]$, it is useful to recall some Sobolev spaces. For $m \in \mathbb{N}$, let $\mathcal{H}^m(\mathbb{R}^d)$ be the domain of the operator $\langle P \rangle^m$ that is

$$\mathcal{H}^m(\mathbb{R}^d) = \{f \in L^2; \langle P \rangle^m f \in L^2\},$$

where $\langle P \rangle^m f$ is computed in Fourier space. Note that $\mathcal{H}^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Moreover the dual of $\mathcal{H}^m(\mathbb{R}^d)$ can be identified with $\langle P \rangle^{-m} L^2(\mathbb{R}^d)$.

The form $[H, A]$ is defined on $\mathcal{D}(H) \cap \mathcal{D}(A)$ by $\langle f, [H, iA]g \rangle = \langle Hf, Af \rangle - \langle Af, Hf \rangle$. Let $\chi_c \in \mathcal{C}_c^\infty(\mathbb{R}^d; \mathbb{R})$ such that $\chi_c = 1$ on the compact support of V_c . By statement (1) in Lemma 3.1, the form $[H, A]$ coincides, on $\mathcal{D}(\langle P \rangle \langle Q \rangle) \cap \mathcal{D}(H_0)$, with the form $[H, iA]'$ given by

$$(3.1) \quad \begin{aligned} \langle f, [H, iA]'g \rangle &= \langle f, [H_0, iA]'g \rangle + \langle f, [V_{sr}(Q), iA]'g \rangle + \langle f, [V_c(Q), iA]'g \rangle \\ &\quad + \langle f, [V_{lr}(Q), iA]'g \rangle + \langle f, [W_{\alpha\beta}(Q), iA]'g \rangle \end{aligned}$$

where $\langle f, [H_0, iA]'g \rangle = \langle f, 2H_0g \rangle$, $\langle f, [V_{lr}, iA]'g \rangle = -\langle f, Q \cdot (\nabla V_{lr})(Q)g \rangle$,

$$\begin{aligned} \langle f, [V_{sr}(Q), iA]'g \rangle &= \langle V_{sr}(Q)Qf, iPg \rangle + \langle iPf, V_{sr}(Q)Qg \rangle \\ &\quad + d\langle f, V_{sr}(Q)g \rangle, \\ \langle f, [V_c(Q), iA]'g \rangle &= \langle V_c(Q)f, \chi_c(Q)Q \cdot iPg \rangle + \langle \chi_c(Q)Q \cdot iPf, V_c(Q)g \rangle \\ &\quad + d\langle f, V_c(Q)g \rangle, \\ \langle f, [W_{\alpha\beta}(Q), iA]'g \rangle &= \langle W_{\alpha\beta}(Q)Qf, iPg \rangle + \langle iPf, W_{\alpha\beta}(Q)Qg \rangle \\ &\quad + d\langle f, W_{\alpha\beta}(Q)g \rangle. \end{aligned}$$

Here $\langle V_{sr}(Q)Qf, iPg \rangle$ means $\sum_{j=1}^d \langle V_{sr}(Q)Q_j f, iP_j g \rangle$. For $\alpha = 1$, we still have the usual situation where the potential belongs to $\mathcal{C}^1(A)$ (like in [ABG, GJ2]) but this is not true anymore if $\alpha > 1$, since $[W_{\alpha\beta}(Q), iA]'$ is not bounded. Using (3.1), we see that the forms $[V_{sr}(Q), iA]$, $[V_c(Q), iA]$, and $[V_{lr}(Q), iA]$ are bounded on \mathcal{F} and associated to a compact operator from \mathcal{F} to its dual \mathcal{F}' , for \mathcal{F} given by $\mathcal{H}^1(\mathbb{R}^d)$, $\mathcal{H}^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, respectively.

For any function $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{C})$, the forms $\theta(H)[H, iA]\theta(H)$ and $\theta(H)[H, iA]'\theta(H)$ are defined on $\mathcal{D}(\langle Q \rangle)$, by Lemma 3.1, and coincide.

Assumption 3.2. *Let $\alpha \geq 1 \geq \beta$. Recall that $\beta_{lr} = \min(\beta; \rho_{lr})$. We take β and ρ_{lr} such that $\beta + \beta_{lr} > 1$ or, equivalently, $\beta > 1/2$ and $\rho_{lr} > 1 - \beta$. If $\alpha > 1$, we consider a compact interval \mathcal{J} such that $\mathcal{J} \subset]0; +\infty[$. If $\alpha = 1$, we consider a small enough, compact interval \mathcal{J} such that $\mathcal{J} \subset]0; k^2/4[$.*

Remark 3.3. Assumption 3.2 is identical to Assumption 1.2, except for the change of the name of the interval and for the smallness requirement when $\alpha = 1$. We actually need to work in a slightly larger interval \mathcal{J} than the interval \mathcal{I} considered in Theorem 1.5. In the case $\alpha = 1$, the smallness of \mathcal{J} (and thus of the above \mathcal{I}) is the one that matches the smallness required in Proposition 2.1. It depends only on the distance of the middle point of \mathcal{J} to $k^2/4$.

Proposition 3.4. *Under Assumptions 1.1 and 3.2, let $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{R})$ with support inside $\mathring{\mathcal{J}}$, the interior of \mathcal{J} , the form $\theta(H)[W_{\alpha\beta}(Q), iA]\theta(H)$ extends to a bounded form on $L^2(\mathbb{R}^d)$ that is associated to a compact operator.*

Remark 3.5. In dimension $d = 1$ with $\alpha = 1$, the result still holds true if the function θ is supported inside $]0; +\infty[\setminus \{k^2/4\}$.

Our proof of Proposition 3.4 relies on Propositions 2.1, 2.4, and on the following

Lemma 3.6. *Assume Assumptions 1.1 and 1.2 satisfied. Let $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{C})$. Then $\langle Q \rangle^{\beta_{lr}}(\theta(H) - \theta(H_0))$ and $\langle Q \rangle^{\beta_{lr}}P(\theta(H) - \theta(H_0))$ are bounded on $L^2(\mathbb{R}^d)$.*

Proof. See Lemma C.5. □

Proof of Proposition 3.4. It suffices to study $\theta(H)[W_{\alpha\beta}(Q), iA]'\theta(H)$. Since $\beta > 0$, $\theta(H)W_{\alpha\beta}(Q)\theta(H)$ extends to a bounded form associated to a compact operator. We study the form $(f, g) \mapsto \langle P\theta(H)f, W_{\alpha\beta}(Q)Q\theta(H)g \rangle$, the remaining term being treated in a similar way. We write this form as

$$\begin{aligned}
 \theta(H)P \cdot QW_{\alpha\beta}(Q)\theta(H) &= (\theta(H) - \theta(H_0))P \cdot QW_{\alpha\beta}(Q)(\theta(H) - \theta(H_0)) \\
 &\quad + (\theta(H) - \theta(H_0))P \cdot QW_{\alpha\beta}(Q)\theta(H_0) \\
 (3.2) \quad &\quad + \theta(H_0)P \cdot QW_{\alpha\beta}(Q)(\theta(H) - \theta(H_0)) \\
 &\quad + \theta(H_0)P \cdot QW_{\alpha\beta}(Q)\theta(H_0).
 \end{aligned}$$

Using Lemma 3.6 and the fact that $\beta + \beta_{lr} - 1 > 0$, we see that the first three terms on the r.h.s. of (3.2) extends to a compact operator. So does also the last term, by Proposition 2.1 with $\epsilon = 1 - \beta$, if $\alpha = 1$, and by Proposition 2.4 with $p = 1 - \beta$, if $\alpha > 1$. □

Now, we are in position to prove the Mourre estimate.

Proposition 3.7. *Under Assumptions 1.1 and 3.2, let $\theta \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ with support inside the interior $\mathring{\mathcal{J}}$ of the interval \mathcal{J} . Denote by $c > 0$ the infimum of \mathcal{J} . Then the form $\theta(H)[H, iA]\theta(H)$ extends to a bounded one on $L^2(\mathbb{R}^d)$ and there exists a compact operator K on $L^2(\mathbb{R}^d)$ such that*

$$(3.3) \quad \theta(H)[H, iA]\theta(H) \geq 2c\theta(H)^2 + K.$$

Proof. Let K_0 be the operator associated with the form

$$\begin{aligned}
 &\theta(H)[V_{sr}(Q), iA]\theta(H) + \theta(H)[V_{lr}(Q), iA]\theta(H) \\
 &+ \theta(H)[V_c(Q), iA]\theta(H) + \theta(H)[W_{\alpha\beta}(Q), iA]\theta(H).
 \end{aligned}$$

It is compact by the previous properties and Proposition 3.4. Thus, as forms,

$$\theta(H)[H, iA]\theta(H) = \theta(H)[H_0, iA]\theta(H) + K_0.$$

Since $[H_0, iA] = 2H_0$, the form

$$(\theta(H) - \theta(H_0))[H_0, iA]\theta(H) + \theta(H_0)[H_0, iA](\theta(H) - \theta(H_0))$$

is associated to a compact operator K_1 , by Lemma 3.6, and

$$\begin{aligned}
 \theta(H)[H, iA]\theta(H) &= \theta(H_0)[H_0, iA]\theta(H_0) + K_0 + K_1 \\
 &\geq 2c\theta(H_0)^2 + K_0 + K_1 \\
 &\geq 2c\theta(H)^2 + K_0 + K_1 + K_3,
 \end{aligned}$$

with compact $K_3 = 2c(\theta(H_0)^2 - \theta(H)^2)$. □

4. POLYNOMIAL BOUNDS ON POSSIBLE EIGENFUNCTIONS WITH POSITIVE ENERGY.

In this section, we shall show a polynomially decaying bound on the possible eigenfunctions of H with positive energy. Because of the oscillating behaviour of the potential $W_{\alpha\beta}$, the corresponding result in [FH] does not apply but it turns out that one can adapt the arguments from [FH] to the present situation. We note further that the abstract results in [Ca, CGH] cannot be applied here because of the lack of regularity w.r.t. the generator of dilations.

Proposition 4.1. *Under Assumptions 1.1 and 3.2, let $E \in \mathring{\mathcal{J}}$ and $\psi \in \mathcal{D}(H)$ such that $H\psi = E\psi$. Then, for all $\lambda \geq 0$, $\psi \in \mathcal{D}(\langle Q \rangle^\lambda)$ and $\nabla\psi \in \mathcal{D}(\langle Q \rangle^\lambda)$.*

Corollary 4.2. *Under Assumptions 1.1 and 3.2, for $E \in \mathring{\mathcal{J}}$, $\text{Ker}(H - E) \subset \mathcal{D}(A)$.*

Proof. Let $\psi \in \text{Ker}(H - E)$. By Proposition 4.1, $\nabla\psi \in \mathcal{D}(\langle Q \rangle)$ thus $\psi \in \mathcal{D}(A)$. \square

Proof of Proposition 4.1. We take a function $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{R})$ with support inside $\mathring{\mathcal{J}}$ such that $\theta(E) = 1$. By Proposition 3.7, the Mourre estimate (3.3) holds true.

Now we follow the beginning of the proof of Theorem 2.1 in [FH], making appropriate adaptations. For $\lambda \geq 0$ and $\epsilon > 0$, we consider the function $F : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by $F(x) = \lambda \ln(\langle x \rangle (1 + \epsilon \langle x \rangle)^{-1})$. For all $x \in \mathbb{R}^d$, $\nabla F(x) = g(x)x$ with $g(x) = \lambda \langle x \rangle^{-2} (1 + \epsilon \langle x \rangle)^{-1}$. Let $H(F)$ be the operator defined on the domain $\mathcal{D}(H(F)) := \mathcal{D}(H_0) = \mathcal{H}^2(\mathbb{R}^d)$ by

$$(4.1) \quad H(F) = e^{F(Q)} H e^{-F(Q)} = H - |\nabla F|^2 + (iP \cdot \nabla F + \nabla F \cdot iP).$$

Setting $\psi_F = e^{F(Q)}\psi$, one has $\psi_F \in \mathcal{D}(H_0)$, $H(F)\psi_F = E\psi_F$, and $\langle \psi_F, H\psi_F \rangle = \langle \psi_F, (|\nabla F|^2 + E)\psi_F \rangle$.

Note that, since e^F does not contain decay in $\langle x \rangle$, we a priori need some argument to give a meaning to $\langle \psi_F, [H, iA]\psi_F \rangle$ when $\beta < 1$, because of the contribution of $W_{\alpha\beta}$ in (3.1).

Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{R})$ with $\chi = 1$ near 0 and, for $R \geq 1$, let $\chi_R(t) = \chi(t/R)$. To replace Equation (2.9) in [FH], we claim that

$$(4.2) \quad \lim_{R \rightarrow +\infty} \langle \chi_R(\langle Q \rangle)\psi_F, [H, iA]\chi_R(\langle Q \rangle)\psi_F \rangle = -4 \cdot \|g(Q)^{1/2} A\psi_F\|^2 + \langle \psi_F, G(Q)\psi_F \rangle,$$

where $G : \mathbb{R}^d \ni x \mapsto ((x \cdot \nabla)^2 g)(x) - (x \cdot \nabla |\nabla F|^2)(x)$. Notice that $\chi_R(\langle Q \rangle)\psi_F \in \mathcal{D}(\langle Q \rangle \langle P \rangle)$, so the bracket on the l.h.s. of (4.2) is well defined. Since, for $x \in \mathbb{R}^d$, $|g(x)| \leq \lambda \langle x \rangle^{-1}$ and $|G(x)| = O(\langle x \rangle^{-2})$, so is the r.h.s. By a direct computation,

$$(4.3) \quad \begin{aligned} & 2\Re \langle A\chi_R(\langle Q \rangle)\psi_F, i(H(F) - E)\chi_R(\langle Q \rangle)\psi_F \rangle \\ &= -\langle \chi_R(\langle Q \rangle)\psi_F, [H, iA]\chi_R(\langle Q \rangle)\psi_F \rangle - 4 \cdot \|g(Q)^{1/2} A\chi_R(\langle Q \rangle)\psi_F\|^2 \\ &+ \langle \chi_R(\langle Q \rangle)\psi_F, G(Q)\chi_R(\langle Q \rangle)\psi_F \rangle. \end{aligned}$$

Note that the commutator $[H(F), \chi_R(Q)]_\circ$ is well-defined since $\chi_R(Q)$ preserves the domain of $H(F)$. Furthermore $[H(F), \chi_R(Q)]_\circ = [H_0(F), \chi_R(Q)]_\circ$, where $H_0(F) = e^{F(Q)} H_0 e^{-F(Q)}$ is a pseudodifferential operator. Notice that the l.h.s of (4.3) is given by

$$2\Re \langle A\chi_R(Q)\psi_F, i[H(F), \chi_R(Q)]_\circ \psi_F \rangle.$$

Using an explicit expression for the commutator and the fact that the family of functions $x \mapsto \langle x \rangle \chi'_R(\langle x \rangle)$ is bounded, uniformly w.r.t. R , and converges pointwise to 0, as $R \rightarrow +\infty$, we apply the dominated convergence theorem to see that the l.h.s. of (4.3) tends to 0 and that the last two terms in (4.3) converge to the r.h.s. of (4.2). Thus the limit in (4.2) exists and (4.2) holds true.

Next we claim that

$$\begin{aligned} \lim_{R \rightarrow +\infty} \langle \chi_R(\langle Q \rangle) \psi_F, [H, iA] \chi_R(\langle Q \rangle) \psi_F \rangle &= \langle \theta(H) \psi_F, [H, iA] \theta(H) \psi_F \rangle \\ (4.4) \quad &+ \langle \psi_F, (K_1 B_{1,\epsilon} + B_{2,\epsilon} K_2) \psi_F \rangle \end{aligned}$$

where, on $L^2(\mathbb{R}^d)$, K_1, K_2 are ϵ -independent compact operators and $B_{1,\epsilon}, B_{2,\epsilon}$ are bounded operators satisfying $\|B_{1,\epsilon}\| + \|B_{2,\epsilon}\| = O(\epsilon^0)$. Notice that, by Proposition 3.7, the first term on the r.h.s of (4.4) is well defined and equal to

$$\lim_{R \rightarrow +\infty} \langle \theta(H) \chi_R(\langle Q \rangle) \psi_F, [H, iA] \theta(H) \chi_R(\langle Q \rangle) \psi_F \rangle.$$

Writing each $\chi_R(\langle Q \rangle) \psi_F$ as $\chi_R(\langle Q \rangle) \psi_F = \theta(H) \chi_R(Q) + (1 - \theta(H)) \chi_R(\langle Q \rangle) \psi_F$, we split $\langle \chi_R(\langle Q \rangle) \psi_F, [H, iA] \chi_R(\langle Q \rangle) \psi_F \rangle$ into four terms, one of them tending to the first term on the r.h.s of (4.4). We focus on the others. Since $(1 - \theta(H)) \psi = 0$,

$$\begin{aligned} (1 - \theta(H)) \chi_R(\langle Q \rangle) \psi_F &= -[\theta(H), \chi_R(\langle Q \rangle)]_\circ \psi_F \\ (4.5) \quad &- \chi_R(\langle Q \rangle) [\theta(H), e^{F(Q)}]_\circ \psi, \end{aligned}$$

$$\begin{aligned} P(1 - \theta(H)) \chi_R(\langle Q \rangle) \psi_F &= -P[\theta(H), \chi_R(\langle Q \rangle)]_\circ \psi_F \\ (4.6) \quad &- [P, \chi_R(\langle Q \rangle)] [\theta(H), e^{F(Q)}]_\circ \psi \\ &- \chi_R(\langle Q \rangle) P[\theta(H), e^{F(Q)}]_\circ \psi. \end{aligned}$$

Lemma 4.3. *Recall that $\beta_{lr} = \min(\beta; \rho_{lr}) \leq 1$.*

(1) *For $\sigma \in [0; 1]$, the operators*

$$\langle Q \rangle^{1-\sigma} [\theta(H), e^{F(Q)}]_\circ e^{-F(Q)} \langle Q \rangle^\sigma \quad \text{and} \quad \langle Q \rangle^{1-\sigma} P[\theta(H), e^{F(Q)}]_\circ e^{-F(Q)} \langle Q \rangle^\sigma$$

are bounded on $L^2(\mathbb{R}^d)$, uniformly w.r.t. $\epsilon \in]0; 1]$.

(2) *For $R \geq 1$, the operators*

$$\langle Q \rangle^{1-\beta_{lr}} [\theta(H), \chi_R(\langle Q \rangle)]_\circ \quad \text{and} \quad \langle Q \rangle^{1-\beta_{lr}} P[\theta(H), \chi_R(\langle Q \rangle)]_\circ$$

are bounded on $L^2(\mathbb{R}^d)$ and their norm are $O(R^{-\beta_{lr}})$.

Proof. For the result (2), see the proof of Lemma C.6.

Let us prove (1). Making use of Helffer-Sjöstrand formula (C.5) and of (C.12), for $H' = H$, we can show by induction that, for all $j \in \mathbb{N}^*$,

$$(4.7) \quad \langle Q \rangle^{1-\sigma} \cdot \text{ad}_{\langle Q \rangle}^j(\theta(H)) \cdot \langle Q \rangle^\sigma$$

is bounded on $L^2(\mathbb{R}^d)$. Note that the function e^F can be written as $\varphi_\epsilon(\langle \cdot \rangle)$, where φ_ϵ stays in a bounded set in \mathcal{S}^λ , when ϵ varies in $]0; 1]$. Since $\theta(H) \in \mathcal{C}^\infty(\langle Q \rangle)$ (cf. Lemma 3.1), we can apply Propositions C.3 with $B = \theta(H)$ and $k > \lambda + 1$. By (4.7), the first terms are all bounded on $L^2(\mathbb{R}^d)$. Let us focus on the last one, that contains an integral. Exploiting (C.2) with $\ell = k + 1$, (C.3), (C.7), (4.7), and the fact that $\varphi_\epsilon(\langle \cdot \rangle)$ is bounded below by $1/2$ for $\epsilon \in]0; 1]$, we see that the last term is also bounded on $L^2(\mathbb{R}^d)$. This shows the boundedness of the first operator in (1). For the second one, we can follow the same lines. \square

Proof of Proposition 4.1 continued. Using Lemma 4.3 and (4.5), we get that

$$\begin{aligned} & \lim_{R \rightarrow +\infty} \langle \theta(H) \chi_R(\langle Q \rangle) \psi_F, P \cdot Q W_{\alpha\beta}(Q) (1 - \theta(H)) \chi_R(\langle Q \rangle) \psi_F \rangle \\ &= - \langle K \psi_F, W_{\alpha\beta}(Q) \langle Q \rangle^\beta Q \langle Q \rangle^{-1} \cdot \langle Q \rangle [\theta(H), e^{F(Q)}]_\circ e^{-F(Q)} \psi_F \rangle \end{aligned}$$

where K is an ϵ -independent vector of compact operators and the bounded operator acting on the right ψ_F is uniformly bounded w.r.t. ϵ . Similarly, using Lemma 4.3 and (4.6), we see that

$$\begin{aligned} & \lim_{R \rightarrow +\infty} \langle \theta(H) \chi_R(\langle Q \rangle) \psi_F, W_{\alpha\beta}(Q) Q \cdot P (1 - \theta(H)) \chi_R(\langle Q \rangle) \psi_F \rangle \\ &= - \langle K' \psi_F, W_{\alpha\beta}(Q) \langle Q \rangle^\beta Q \langle Q \rangle^{-1} \cdot \langle Q \rangle P [\theta(H), e^{F(Q)}]_\circ e^{-F(Q)} \psi_F \rangle \end{aligned}$$

with K' compact and an uniformly bounded operator acting on the right ψ_F . Using again (4.5) and (4.6), we also get

$$\begin{aligned} & \lim_{R \rightarrow +\infty} \langle (1 - \theta(H)) \chi_R(\langle Q \rangle) \psi_F, W_{\alpha\beta}(Q) Q \cdot P (1 - \theta(H)) \chi_R(\langle Q \rangle) \psi_F \rangle \\ &= \langle \langle Q \rangle^{-\beta/2} [\theta(H), e^{F(Q)}]_\circ e^{-F(Q)} \langle Q \rangle^{\beta/2} \langle P \rangle K'' \psi_F, \\ & \quad W_{\alpha\beta}(Q) \langle Q \rangle^\beta Q \langle Q \rangle^{-\beta/2} P [\theta(H), e^{F(Q)}]_\circ e^{-F(Q)} \psi_F \rangle \end{aligned}$$

with compact $K'' = \langle P \rangle^{-1} \langle Q \rangle^{-\beta/2}$ and uniformly bounded operators acting on the right ψ_F and on $K'' \psi_F$.

In a similar way, we can treat the last term in the contribution of $[W_{\alpha\beta}(Q), iA]'$ and the contribution of $[V_c(Q), iA]'$, $[V_{sr}(Q), iA]'$, and $[H_0, iA]'$ (cf. (3.1)). This ends the proof of (4.4), yielding, together with (4.2),

$$(4.8) \quad \langle \theta(H) \psi_F, [H, iA] \theta(H) \psi_F \rangle = -4 \cdot \|g(Q)^{1/2} A \psi_F\|^2 + \langle \psi_F, G(Q) \psi_F \rangle - \langle \psi_F, (K_1 B_{1,\epsilon} + B_{2,\epsilon} K_2) \psi_F \rangle.$$

Assume that, for some $\lambda > 0$, $\psi \notin \mathcal{D}(\langle Q \rangle^\lambda)$. We define $\Psi_\epsilon = \|\psi_F\|^{-1} \psi_F$. As in [FH], $(H_0 + 1) \Psi_\epsilon$ and thus Ψ_ϵ both go to 0, weakly in $L^2(\mathbb{R}^d)$, as $\epsilon \rightarrow 0$. Therefore $\|K_1 \Psi_\epsilon\| + \|K_2 \Psi_\epsilon\| \rightarrow 0$, as $\epsilon \rightarrow 0$. Since $G(Q)(H_0 + 1)^{-1}$ is compact, $\|G(Q) \Psi_\epsilon\| \rightarrow 0$. Since $(1 - \theta(H)) \psi = 0$,

$$(1 - \theta(H)) \Psi_\epsilon = [\theta(H), e^{F(Q)}]_\circ e^{-F(Q)} \langle Q \rangle \langle Q \rangle^{-1} (H_0 + 1)^{-1} (H_0 + 1) \Psi_\epsilon.$$

Since $[\theta(H), e^{F(Q)}]_\circ e^{-F(Q)} \langle Q \rangle$ is uniformly bounded w.r.t. ϵ , by Lemma 4.3, and $\langle Q \rangle^{-1} (H_0 + 1)^{-1}$ is compact, the weak convergence to 0 of $(H_0 + 1) \Psi_\epsilon$ implies the norm convergence to 0 of $(1 - \theta(H)) \Psi_\epsilon$. Thus $\lim_{\epsilon \rightarrow 0} \|\theta(H) \Psi_\epsilon\| = 1$.

Dividing by $\|\psi_F\|^2$ in (4.8) and then taking the “ $\liminf_{\epsilon \rightarrow 0}$ ”, we get

$$\liminf_{\epsilon \rightarrow 0} \langle \theta(H) \Psi_\epsilon, [H, iA] \theta(H) \Psi_\epsilon \rangle = -4 \cdot \liminf_{\epsilon \rightarrow 0} \|g(Q)^{1/2} A \Psi_\epsilon\|^2 \leq 0.$$

Now, we apply the Mourre estimate (3.3) to Ψ_ϵ , yielding

$$\liminf_{\epsilon \rightarrow 0} \langle \theta(H) \Psi_\epsilon, [H, iA] \theta(H) \Psi_\epsilon \rangle \geq 2c \liminf_{\epsilon \rightarrow 0} \|\theta(H) \Psi_\epsilon\|^2 + 0 = 2c > 0$$

and a contradiction. Therefore $\psi \in \mathcal{D}(\langle Q \rangle^\lambda)$, for all $\lambda > 0$.

Take $\lambda > 0$. Since $V(Q)$ is H_0 -bounded with relative bound 0, we can find, for any $\delta \in]0, 1[$, some $C_\delta > 0$ such that, for all $\epsilon > 0$,

$$|\langle \psi_F, V(Q) \psi_F \rangle| \leq \delta \langle \psi_F, H_0 \psi_F \rangle + C \|\psi_F\|^2 = \delta \|\nabla \psi_F\|^2 + C \|\psi_F\|^2.$$

Using the equality $\langle \psi_F, H\psi_F \rangle = \langle \psi_F, (|\nabla F|^2(Q) + E)\psi_F \rangle$, we can find $C', C'' > 0$ such that, for all $\epsilon > 0$,

$$(4.9) \quad \|\nabla \psi_F\|^2 \leq C' \|\psi_F\|^2 \leq C' \|\langle Q \rangle^\lambda \psi\|^2 =: (C'')^2.$$

Now, $\nabla \psi_F = (\nabla F)(Q)\psi_F + e^{F(Q)}\nabla \psi$, yielding, for all $\epsilon > 0$,

$$\|e^{F(Q)}\nabla \psi\| \leq C'' + \|\psi_F\| \leq C'' + \|\langle Q \rangle^\lambda \psi\|.$$

This shows that $\nabla \psi$ belongs to $\mathcal{D}(\langle Q \rangle^\lambda)$. \square

5. LOCAL FINITNESS OF THE POINT SPECTRUM.

In the usual Mourre theory, one easily deduces from a Mourre estimate on some compact interval \mathcal{J} the finitness of the point spectrum in any compact interval $\mathcal{I} \subset \mathring{\mathcal{J}}$, the interior of \mathcal{J} , thanks to the virial Theorem. In the present situation, we do not know if we have the required regularity of H w.r.t. A to apply the abstract virial Theorem. But, thanks to Corollary 4.2, we are able to get it in a trivial way.

Proposition 5.1. *Under Assumptions 1.1 and 3.2, let $E \in \mathring{\mathcal{J}}$ and $\psi \in \mathcal{D}(H)$ such that $H\psi = E\psi$. Then $\langle \psi, [H, A]\psi \rangle = 0$.*

Proof. Since $\psi \in \mathcal{D}(A)$ by Corollary 4.2, $\langle \psi, [H, A]\psi \rangle$ is well defined and

$$\langle \psi, [H, A]\psi \rangle = \langle H\psi, A\psi \rangle - \langle A\psi, H\psi \rangle = 0,$$

because E is real and A is self-adjoint. \square

Now, the Mourre estimate in Proposition 3.7 gives the

Corollary 5.2. *Under Assumptions 1.1 and 3.2, for any compact interval $\mathcal{I} \subset \mathring{\mathcal{J}}$, the point spectrum of H inside \mathcal{I} is finite (counted with multiplicity).*

Proof. One can follow the usual proof. See [ABG] p. 295 or [Mo], for instance. \square

Thanks to Corollaries 4.2 and 5.2, we are able to prove the following regularity result. The precise definition of the mentioned regularity is given in Appendix B.

Corollary 5.3. *Under Assumptions 1.1 and 3.2, for any $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{C})$ with support included in $\mathring{\mathcal{J}}$, $\theta(H)\Pi \in \mathcal{C}^1(A)$ and $\theta(H)\Pi \in \mathcal{C}^\infty(\langle Q \rangle)$.*

Proof. For $\psi \in \mathcal{D}(A)$, the projector $\langle \psi, \cdot \rangle \psi$ belongs to $\mathcal{C}^1(A)$ since the form

$\mathcal{D}(A)^2 \ni (\varphi_1; \varphi_2) \mapsto \langle \varphi_1, [\langle \psi, \cdot \rangle \psi, A]\varphi_2 \rangle = \overline{\langle \psi, \varphi_1 \rangle} \langle A\psi, \varphi_2 \rangle - \langle \psi, \varphi_2 \rangle \langle \varphi_1, A\psi \rangle$ extends to a bounded one. By Corollary 5.2, the point spectrum of H inside the support of θ is some $\{\lambda_1; \dots; \lambda_n\}$ and there exist $\psi_1, \dots, \psi_n \in \mathcal{D}(H)$ such that $H\psi_j = \lambda_j\psi_j$, for all j . By Corollary 4.2, $\psi_j \in \mathcal{D}(A)$, for all j . Since

$$(5.1) \quad \theta(H)\Pi = \sum_{j=1}^n \theta(\lambda_j) \langle \psi_j, \cdot \rangle \psi_j,$$

$\theta(H)\Pi \in \mathcal{C}^1(A)$.

Similarly, we show $\theta(H)\Pi \in \mathcal{C}^\infty(\langle Q \rangle)$ using (5.1) and Proposition 4.1. \square

6. EXPONENTIAL BOUNDS ON POSSIBLE EIGENFUNCTIONS WITH POSITIVE ENERGY.

In this section, we take $\alpha > 1$, consider positive energies and show that, a possible eigenfunction of H , associated to such energies, must satisfy some exponential bound in the L^2 -norm. The result and the proof are almost identical to Theorem 2.1 in [FH] and its proof. We only change some argument to take into account the influence of our oscillating potential. We try to explain in Remark 6.2 below why we do not treat here the case $\alpha = 1$. However, we have some information at high energy in the case $\alpha = \beta = 1$ (see Remark 6.3).

Proposition 6.1. *Under Assumptions 1.1 and 1.2 with $\alpha > 1$, let $E > 0$ and $\psi \in \mathcal{D}(H)$ such that $H\psi = E\psi$. Let*

$$r = \sup \left\{ t^2 + E ; t \in [0; +\infty[\quad \text{and} \quad e^{t\langle Q \rangle} \psi \in L^2(\mathbb{R}^d) \right\} \geq E.$$

Then $r = +\infty$.

Proof. We exactly follow the lines of the last part of the proof of Theorem 2.1 in [FH], except for one important argument and some details. Just after formula (2.35) in [FH], the authors use the boundedness of $(H_0 + 1)^{-1}[H, iA](H_0 + 1)^{-1}$ to show that the l.h.s. of this formula (2.35) is bounded w.r.t. λ . Here we cannot do so (the previous form is actually unbounded) but provide another argument (see (6.4)) to get the same conclusion. For completeness, we recall the main lines of the last part of the proof of Theorem 2.1 in [FH].

Assume that the result is false. Then r is finite. By Proposition 3.7, the Mourre estimate (3.3) holds true for any $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$ with small enough support around r . Let us take such a function θ that is also identically 1 on some open interval \mathcal{I}' centered at r . If $r = E$, let $r_0 = r = E$, else let $r_0 < r$ such that $r_0 \in \mathcal{I}'$. We set $r_0 = t_0^2 + E$ with $t_0 \geq 0$. We take $t_1 > 0$ such that $r_1 := (t_0 + t_1)^2 + E > r$ and $r_1 \in \mathcal{I}'$. We may assume that $t_1 \leq 1$.

For $\lambda \geq 0$, let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be defined by $F(x) = t_0 \langle x \rangle + \lambda \ln(1 + t_1 \lambda^{-1} \langle x \rangle)$. By the definition of r , we know that $\langle Q \rangle^\lambda e^{t_0 \langle Q \rangle} \psi \in L^2(\mathbb{R}^d)$ (if $r = E$ i.e. $t_0 = 0$, this follows from Proposition 4.1). Thus ψ belongs to the domain of the multiplication operator $e^{F(Q)}$. We define $\psi_F = e^{F(Q)} \psi$ and $\Psi_\lambda = \|\psi_F\|^{-1} \psi_F$. By the end of the proof of Proposition 4.1, we can show that $\nabla \psi_F$ belongs to the domain of $\langle Q \rangle$. Thus $\psi_F \in \mathcal{D}(A)$, therefore the expectation value $\langle \psi_F, [H, iA] \psi_F \rangle$ is well defined, and a direct computation gives

$$(6.1) \quad \langle \psi_F, [H, iA] \psi_F \rangle = -4 \cdot \|g(Q)^{1/2} A \psi_F\|^2 + \langle \psi_F, G(Q) \psi_F \rangle,$$

where g is defined by $F(x) = g(x)x$ and $G(x) = ((Q.P)^2 g)(x) - (Q.P |\nabla F|^2)(x)$. Uniformly w.r.t. $\lambda \geq 1$, $|\nabla F(x)| = O(\langle x \rangle^0)$ and the matrix norm $|(\nabla \otimes \nabla) F(x)| = O(\langle x \rangle^{-1})$. Notice that $e^{(t_0+t_1)\langle Q \rangle} \psi \notin L^2(\mathbb{R}^d)$. As in [FH], we can show that $\lambda \mapsto \Psi_\lambda$, $\lambda \mapsto \nabla \Psi_\lambda$, and $\lambda \mapsto H_0 \Psi_\lambda$ are continuous for the $L^2(\mathbb{R}^d)$ -norm and tend to 0 weakly in $L^2(\mathbb{R}^d)$, as $\lambda \rightarrow +\infty$. This implies, in particular, that, for any $\delta > 0$,

$$(6.2) \quad \lim_{\lambda \rightarrow +\infty} \|\langle Q \rangle^{-\delta} \Psi_\lambda\| = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow +\infty} \|\langle Q \rangle^{-\delta} \nabla \Psi_\lambda\| = 0.$$

Since $|G(x)| = O(\langle x \rangle^{-1}) + t_1(t_0 + t_1)$, uniformly w.r.t. $\lambda \geq 1$, we derive from (6.1) and (6.2) that

$$(6.3) \quad \limsup_{\lambda \rightarrow +\infty} \langle \Psi_\lambda, [H, iA] \Psi_\lambda \rangle \leq t_1(t_0 + t_1).$$

Now, we claim that

$$(6.4) \quad \sup_{\lambda \geq 1} |\langle \Psi_\lambda, [H, iA] \Psi_\lambda \rangle| < +\infty.$$

Thanks to (6.4), we can follow the arguments of [FH] to get the desired contradiction for small enough t_1 .

We are left with the proof of (6.4). We note that the form $\langle P \rangle^{-1} [H - W_{\alpha\beta}, iA] \langle P \rangle^{-1}$ extends to a bounded one. Together with the above properties of the family $(\Psi_\lambda)_\lambda$, this shows the boundedness of $|\langle \Psi_\lambda, [H - W_{\alpha\beta}, iA] \Psi_\lambda \rangle|$ w.r.t. λ . To get the boundedness of the rest, namely $|\langle \Psi_\lambda, [W_{\alpha\beta}, iA] \Psi_\lambda \rangle|$, it suffices to show

$$(6.5) \quad \sup_{\lambda \geq 1} |\langle W_{\alpha\beta} \Psi_\lambda, Q \cdot P \Psi_\lambda \rangle| < +\infty.$$

To this end, we shall use the fact that ψ is localized w.r.t. H at energy E and “move” this property through the $e^{F(Q)}$ factors appearing in the scalar product appearing in (6.5).

Since $V(Q)$ is H_0 -compact, there exists some $c_0 > 0$ such that $H \geq -c_0$. For $m > c_0$, $m + H$ is invertible with bounded inverse. Recall that $H(F)$ is defined in (4.1). Let $H_0(F) = e^{F(Q)} H_0 e^{-F(Q)}$. Since $|\nabla F(x)| = O(\langle x \rangle^0)$, uniformly w.r.t. $\lambda \geq 1$, we can find $m > 0$ large enough such that, for all $\lambda \geq 1$, $m + H(F)$ and $m + H_0(F)$ are invertible with uniformly bounded inverse. Moreover, we see that $V(Q)(m + H(F))^{-1}$ and $V(Q)(m + H_0(F))^{-1}$ are uniformly bounded.

For $\lambda \geq 1$, F stays in a bounded set of the symbol class $S(1; g)$ (see Appendix A for details). Thus, by pseudodifferential calculus, $\langle P \rangle^2 (m + H_0(F))^{-1}$ is uniformly bounded. By the resolvent formula, so is also $\langle P \rangle^2 (m + H(F))^{-1}$.

Since $H_0 \in \mathcal{C}^1(\langle Q \rangle)$ and $H \in \mathcal{C}^1(\langle Q \rangle)$ by Lemma 3.1, since F is smooth, $H_0(F) \in \mathcal{C}^1(\langle Q \rangle)$ and $H(F) \in \mathcal{C}^1(\langle Q \rangle)$. Using Propositions C.3 and C.4, we see that, for $\epsilon \in [0, 1]$, $\langle Q \rangle^\epsilon (m + H_0(F))^{-1} \langle Q \rangle^{-\epsilon}$ and $\langle Q \rangle^\epsilon (m + H(F))^{-1} \langle Q \rangle^{-\epsilon}$ are bounded, uniformly w.r.t. $\lambda \geq 1$.

For $\ell \in \mathbb{N}$, we can write $\psi = (m + E)^\ell (m + H)^{-\ell} \psi$. By a direct computation,

$$e^{F(Q)} (m + H)^{-1} e^{-F(Q)} = (m + H(F))^{-1}.$$

Thus, for $\ell_1, \ell_2 \in \mathbb{N}$,

$$(6.6) \quad \begin{aligned} & \langle W_{\alpha\beta} \Psi_\lambda, Q \cdot P \Psi_\lambda \rangle \\ &= (m + E)^{\ell_1 + \ell_2} \langle Q W_{\alpha\beta} (m + H(F))^{-\ell_1} \Psi_\lambda, P (m + H(F))^{-\ell_2} \Psi_\lambda \rangle. \end{aligned}$$

In (6.6), we write

$$\begin{aligned} (m + H(F))^{-\ell_1} &= \left((m + H_0(F))^{-1} - (m + H_0(F))^{-1} V(Q) (m + H(F))^{-1} \right)^{\ell_1}, \\ (m + H(F))^{-\ell_2} &= \left((m + H_0(F))^{-1} + (m + H(F))^{-1} V(Q) (m + H_0(F))^{-1} \right)^{\ell_2}, \end{aligned}$$

and expand the products. The expansion contains, up to the factor $(m + E)^{\ell_1 + \ell_2}$, terms of the form

$$(6.7) \quad \langle Q W_{\alpha\beta} (m + H_0(F))^{-1} V(Q) (m + H(F))^{-1} B_1 \Psi_\lambda, B_2 \Psi_\lambda \rangle,$$

where B_1 and B_2 are uniformly bounded operators. By Assumption 1.2, $\langle Q \rangle^{1-\beta-\beta_{lr}}$ is bounded. Since, by the resolvent formula,

$$\begin{aligned} & \langle Q \rangle^{\beta_{lr}} V(Q) (m + H(F))^{-1} \\ &= \langle Q \rangle^{\beta_{lr}} (V(Q) - V_c(Q)) (m + H(F))^{-1} \\ & \quad + \langle Q \rangle^{\beta_{lr}} \chi_c(Q) V_c(Q) \langle P \rangle^{-2} \langle P \rangle^2 (m + H_0(F))^{-1} \\ & \quad - \langle Q \rangle^{\beta_{lr}} \chi_c(Q) V_c(Q) \langle P \rangle^{-2} \langle P \rangle^2 (m + H_0(F))^{-1} V(m + H(F))^{-1}, \end{aligned}$$

the operator $\langle Q \rangle^{\beta_{lr}} V(Q) (m + H(F))^{-1}$ is uniformly bounded. Therefore all the terms of the form (6.7) are bounded. Up to the factor $(m + E)^{\ell_1 + \ell_2}$, the previous expansion contains also terms of the form

$$(6.8) \quad \langle Q W_{\alpha\beta} B'_1 \Psi_\lambda, P(m + H(F))^{-1} V(Q) (m + H_0(F))^{-1} B'_2 \Psi_\lambda \rangle,$$

for uniformly bounded operators B'_1 and B'_2 . We note that $\langle Q \rangle^{\beta_{lr}} P \langle Q \rangle^{-\beta_{lr}} \langle P \rangle^{-1}$ is bounded and that $\langle Q \rangle^{\beta_{lr}} \langle P \rangle^1 (m + H(F))^{-1} \langle Q \rangle^{-\beta_{lr}}$ is uniformly bounded, use again the above arguments to conclude that all the terms of the form (6.8) are bounded functions of λ . We are left with the term

$$(m + E)^{\ell_1 + \ell_2} \langle Q W_{\alpha\beta} (m + H_0(F))^{-\ell_1} \Psi_\lambda, P(m + H_0(F))^{-\ell_2} \Psi_\lambda \rangle.$$

By pseudodifferential calculus,

$$\langle P \rangle^{2\ell_1} (m + H_0(F))^{-\ell_1} \quad \text{and} \quad \langle P \rangle^{2\ell_2 - 1} P(m + H_0(F))^{-\ell_2}$$

are uniformly bounded. Thus, by Proposition 2.4, this last term is bounded, if we choose ℓ_1 and ℓ_2 large enough. This proves (6.5) and therefore (6.4). \square

Remark 6.2. In the above proof, we used the assumption $\alpha > 1$ to get (6.4). Indeed, we managed to move a "localisation" $(m + H)^{-\ell}$ through the multiplication operator $e^{F(Q)}$, creating in this way the factors $\langle P \rangle^{-\ell_1}$ and $\langle P \rangle^{-\ell_2}$. Then we applied Proposition 2.4 that only holds true for $\alpha > 1$ (see Remark 2.5). In the case $\alpha = 1$, it is natural to try to move an appropriate localisation $\theta(H)$ through $e^{F(Q)}$ and then use Proposition 2.1. We do not know how to bound the operator $e^{F(Q)} \theta(H) e^{-F(Q)}$ uniformly w.r.t. λ , when θ is smooth and compactly supported. Formally, $e^{F(Q)} \theta(H) e^{-F(Q)} = \theta(H(F))$ where $H(F) = e^{F(Q)} H e^{-F(Q)}$, but the latter is not self-adjoint (see (4.1)).

Remark 6.3. In the case $\alpha = \beta = 1$, the Mourre estimate is valid at high energy, say on any compact interval included in some $[a; +\infty[$ with $a > 0$ (cf. the proof of Proposition 3.7). Take an energy $E > a$ and $\psi \in \mathcal{D}(H)$ such that $H\psi = E\psi$. The proof of Theorem 2.1 in [FH] works in this situation and yields the conclusion of Proposition 6.1, namely $r = +\infty$.

7. EIGENFUNCTIONS CANNOT SATISFY UNLIMITED EXPONENTIAL BOUNDS.

In this section, we require some condition on the form $[V_c, iA]$ and Assumption 1.1 with $\beta \geq 1/2$. We study the states $\psi \in \mathcal{D}(H)$ such that $H\psi = E\psi$, for some $E \in \mathbb{R}$, and ψ belongs to the domain of the multiplication operator $e^{\gamma\langle Q \rangle}$, for all $\gamma \geq 0$. We shall show that such ψ must be zero. Our proof is inspired by the corresponding result in [FHHH2] (see also Theorem 4.18 in [CFKS]). The key new argument is an appropriate bound on the contribution of the oscillating potential $W_{\alpha\beta}$ to the

commutator form $[H, iA]$ and is easily granted by the assumption $\beta \geq 1/2$. This provides in particular a proof of Theorem 1.11.

Together with Assumption 1.1 with $\beta \geq 1/2$, we require further, as in [FHHH2], that the form $[V_c, iA]$ is H_0 -form-lower-bounded with relative bound less than 2. Precisely, we demand that

$$(7.1) \quad \begin{aligned} \exists \epsilon_c > 0, \exists \lambda_c > 0; \forall \varphi \in \mathcal{D}(H) \cap \mathcal{D}(A), \\ \langle \varphi, [V_c, iA]\varphi \rangle \geq (\epsilon_c - 2)\langle \varphi, H_0\varphi \rangle - \lambda_c \|\varphi\|^2. \end{aligned}$$

We shall need the following known

Lemma 7.1. *Under the previous assumptions,*

$$(7.2) \quad \begin{aligned} \forall \delta \in]0; 1[, \exists \mu_\delta > 0; \forall \varphi \in \mathcal{D}(H) \cap \mathcal{D}(A), \\ \langle \varphi, H_0\varphi \rangle \geq \delta \langle \varphi, H\varphi \rangle - \mu_\delta \|\varphi\|^2. \end{aligned}$$

$$(7.3) \quad \begin{aligned} \forall \epsilon > 0, \exists \lambda_\epsilon > 0; \forall \varphi \in \mathcal{D}(H) \cap \mathcal{D}(A), \\ \langle \varphi, [H - W_{\alpha\beta}, iA]\varphi \rangle \geq (\epsilon_c - \epsilon)\langle \varphi, H_0\varphi \rangle - \lambda_\epsilon \|\varphi\|^2. \end{aligned}$$

Proof. Since $V(Q)$ is H_0 -compact, it is H_0 -bounded with relative bound 0. This implies (7.2) (see [K]). Let $\epsilon > 0$. Since $Q \cdot \nabla V_{lr}(Q)$ is H_0 -compact, we can similarly find some $\lambda_1 > 0$ such that, for all $\varphi \in \mathcal{D}(H) \cap \mathcal{D}(A)$,

$$\langle \varphi, [V_{lr}(Q), iA]\varphi \rangle \geq -\epsilon 2^{-1} \langle \varphi, H_0\varphi \rangle - \lambda_1 \|\varphi\|^2.$$

Since $Q \cdot V_{sr}(Q)$ is H_0 -compact, we can find some $\lambda_2 > 0$ such that, for all $\varphi \in \mathcal{D}(H) \cap \mathcal{D}(A)$,

$$|\langle \varphi, [V_{sr}(Q), iA]\varphi \rangle| \leq (\epsilon 4^{-1} \|\nabla \varphi\| + 2\lambda_2 \|\varphi\|) \cdot \|\nabla \varphi\|.$$

Using $2\|\varphi\| \cdot \|\nabla \varphi\| \leq \eta \|\nabla \varphi\|^2 + \eta^{-1} \|\varphi\|^2$, for $\eta = \epsilon(4\lambda_2)^{-1} > 0$, we get, for some $\lambda_3 > 0$, for all $\varphi \in \mathcal{D}(H) \cap \mathcal{D}(A)$,

$$\langle \varphi, [V_{sr}(Q), iA]\varphi \rangle \geq -\epsilon 2^{-1} \langle \varphi, H_0\varphi \rangle - \lambda_3 \|\varphi\|^2.$$

Summing up these properties and using (7.1), we obtain (7.3). \square

As in Section 6, we shall use a conjugaison by an appropriate $e^{F(Q)}$. For $\gamma > 0$, let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be the smooth function defined by $F(x) = \gamma \langle x \rangle$. Setting $g(x) = \gamma \langle x \rangle^{-1}$, $\nabla F(x) = g(x)x$ and

$$(7.4) \quad |\nabla F(x)|^2 = \gamma^2 (1 - \langle x \rangle^{-2}).$$

A direct computation gives

$$(7.5) \quad ((Q \cdot P)^2 g)(x) = \gamma \langle x \rangle^{-1} (1 - \langle x \rangle^{-2}) (1 - 3 \langle x \rangle^{-2}),$$

$$(7.6) \quad -((Q \cdot P)(|\nabla F|^2))(x) = -2\gamma^2 \langle x \rangle^{-2} (1 - \langle x \rangle^{-2}) \leq 0.$$

Proposition 7.2. *Under Assumption 1.1 with $\beta \geq 1/2$, and (7.1), take $\psi \in \mathcal{D}(H)$ and $E \in \mathbb{R}$ such that $H\psi = E\psi$. Assume further that, for all $\gamma \geq 0$, ψ belongs to the domain of the multiplication operator $e^{\gamma \langle Q \rangle}$. Then $\psi = 0$.*

Remark 7.3. Note that Proposition 7.2 applies under (7.1) and Assumptions 1.1 and 1.2. In particular, the case $\alpha = 1$ is allowed.

Proof of Proposition 7.2. We always consider $\gamma \geq 1$. By assumption, ψ belongs to the domain of the multiplication operator $e^{F(Q)}$. Setting $\psi_F = e^{F(Q)}\psi$, we claim that

$$(7.7) \quad |\langle \psi_F, [W_{\alpha\beta}(Q), iA]\psi_F \rangle| \leq \|g(Q)^{1/2}A\psi_F\|^2 + \gamma^{-1}\|\psi_F\|^2.$$

From the definition of the form $[W_{\alpha\beta}(Q), iA]$, we observe that

$$\begin{aligned} |\langle \psi_F, [W_{\alpha\beta}(Q), iA]\psi_F \rangle| &\leq 2 \cdot \|g(Q)^{1/2}A\psi_F\| \cdot \|g(Q)^{-1/2}\langle Q \rangle^{-\beta}\psi_F\| \\ &\leq 2 \cdot \|g(Q)^{1/2}A\psi_F\| \cdot \gamma^{-1/2} \cdot \|\langle Q \rangle^{1/2-\beta}\psi_F\| \\ &\leq 2 \cdot \|g(Q)^{1/2}A\psi_F\| \cdot \gamma^{-1/2} \cdot \|\psi_F\| \end{aligned}$$

since we assumed that $\beta \geq 1/2$. Now (7.7) follows from the use of the inequality $2ab \leq a^2 + b^2$, for $a, b \geq 0$.

Now, we essentially follow the argument in the proof of Theorem 4.18 in [CFKS] and prove the result by contradiction. Assume that $\psi \neq 0$. Let $\psi_F = e^{F(Q)}\psi$. The formula (6.1) is valid with the new function F . As in the proof of Proposition 4.1, we also have

$$(7.8) \quad \langle \psi_F, H\psi_F \rangle = \langle \psi_F, (|\nabla F|^2(Q) + E)\psi_F \rangle.$$

Combining (6.1) and (7.7), we get, for $\gamma \geq 1$,

$$\begin{aligned} \langle \psi_F, [H - W_{\alpha\beta}(Q), iA]\psi_F \rangle &\leq -3 \cdot \|g(Q)^{1/2}A\psi_F\|^2 + \gamma^{-1}\|\psi_F\|^2 \\ &\quad + \langle \psi_F, G(Q)\psi_F \rangle \\ (7.9) \quad \langle \psi_F, [H - W_{\alpha\beta}(Q), iA]\psi_F \rangle &\leq \langle \psi_F, G(Q)\psi_F \rangle + \gamma^{-1}\|\psi_F\|^2, \end{aligned}$$

where $G(Q) = (Q.P)^2g - (Q.P)(|\nabla F|^2)$. Next we deduce from (7.3) and (7.2) in Lemma 7.1, and (7.8), that, for all $\delta \in]0; \epsilon_c[$, there exist some $\rho_\delta, \rho'_\delta > 0$ such that, for all $\gamma \geq 1$,

$$\begin{aligned} \langle \psi_F, [H - W_{\alpha\beta}(Q), iA]\psi_F \rangle &\geq \delta \langle \psi_F, H_0\psi_F \rangle - \rho_\delta \|\psi_F\|^2 \\ &\geq 2^{-1}\delta (\langle \psi_F, H\psi_F \rangle - 2\mu_{1/2}) - \rho_\delta \|\psi_F\|^2 \\ &\geq 2^{-1}\delta \langle \psi_F, (H - E)\psi_F \rangle - \rho'_\delta \|\psi_F\|^2 \\ (7.10) \quad &\geq 2^{-1}\delta \langle \psi_F, |\nabla F|^2(Q)\psi_F \rangle - \rho'_\delta \|\psi_F\|^2. \end{aligned}$$

In view of (7.4), we introduce the function $f : [0; +\infty[\rightarrow [0; +\infty[$ given by

$$(7.11) \quad f(\gamma) = \langle \psi_F, (1 - \langle Q \rangle^{-2})\psi_F \rangle = \gamma^{-2} \langle \psi_F, |\nabla F|^2(Q)\psi_F \rangle.$$

Since $\psi \neq 0$, we can find $\epsilon > 0$ such that $\|\mathbb{1}_{|\cdot| \geq 2\epsilon}(Q)\psi\| > 0$. For all $\gamma \geq 0$,

$$\frac{\|\mathbb{1}_{|\cdot| \leq \epsilon}(Q)e^{\gamma\langle Q \rangle}\psi\|^2}{\|e^{\gamma\langle Q \rangle}\psi\|^2} \leq \frac{e^{2\gamma\langle \epsilon \rangle}\|\mathbb{1}_{|\cdot| \leq \epsilon}(Q)\psi\|^2}{e^{2\gamma\langle 2\epsilon \rangle}\|\mathbb{1}_{|\cdot| \geq 2\epsilon}(Q)\psi\|^2} \leq e^{2\gamma(\langle \epsilon \rangle - \langle 2\epsilon \rangle)} \frac{\|\psi\|^2}{\|\mathbb{1}_{|\cdot| \geq 2\epsilon}(Q)\psi\|^2}$$

and

$$\begin{aligned} f(\gamma) &\geq (1 - \langle \epsilon \rangle^{-2})\|\mathbb{1}_{|\cdot| \geq \epsilon}(Q)\psi_F\|^2 \\ &\geq (1 - \langle \epsilon \rangle^{-2}) \cdot \left(\|\psi_F\|^2 - \|\mathbb{1}_{|\cdot| \leq \epsilon}(Q)e^{\gamma\langle Q \rangle}\psi\|^2 \right) \\ &\geq (1 - \langle \epsilon \rangle^{-2}) \cdot \|\psi_F\|^2 \cdot \left(1 - C_\epsilon e^{2\gamma(\langle \epsilon \rangle - \langle 2\epsilon \rangle)} \right), \end{aligned}$$

where $C_\epsilon := \|\psi\|^2 \cdot \|\mathbb{1}_{|\cdot| \geq 2\epsilon}(Q)\psi\|^{-2}$. Thus, there exist $C > 0$ and $\Gamma \geq 1$ such that, for $\gamma \geq \Gamma$,

$$(7.12) \quad f(\gamma) \geq C \|\psi_F\|^2 \geq C \|\psi\|^2 > 0.$$

We derive from (7.9) and (7.10), thanks to (7.11) and (7.6), that, for all $\gamma \geq 1$,

$$2^{-1}\delta\gamma^2 f(\gamma) - (\rho'_\delta + \gamma^{-1})\|\psi_F\|^2 \leq \langle \psi_F, G(Q)\psi_F \rangle \leq \langle \psi_F, ((Q \cdot P)^2 g)(Q)\psi_F \rangle.$$

By (7.5), $((Q \cdot P)^2 g)(x) \leq \gamma(1 - \langle x \rangle^{-2})$, for all $x \in \mathbb{R}^d$, yielding, for all $\gamma \geq \Gamma$,

$$\begin{aligned} 2^{-1}\delta\gamma^2 f(\gamma) - (\rho'_\delta + \gamma^{-1})\|\psi_F\|^2 &\leq \gamma f(\gamma) \\ \text{and} \quad (2^{-1}\delta\gamma^2 - \gamma - (\rho'_\delta + \gamma^{-1})C^{-1}) \cdot f(\gamma) &\leq 0, \end{aligned}$$

by (7.12). We get a contradiction for γ large enough. \square

8. LAP AT SUITABLE ENERGIES.

In this section, we prove the limiting absorption principle for H for appropriate energy regions. As already pointed out in [GJ2], one cannot use the usual Mourre theory w.r.t. the generator of dilations A , since the Hamiltonian is not regular enough w.r.t. A . For the same reason, one cannot follow the lines in [Gé]. We were not able to apply the “weighted Mourre theory” developed in [GJ2], which is inspired by [Gé] and is a kind of “localized” Putnam argument. Instead, we follow the more complicated path introduced in [GJ1].

To prepare our result, we need some notation. For $\delta > 0$ and $y \in \mathbb{R}^d$, we set

$$(8.1) \quad g_\delta(y) = (2 - \langle y \rangle^{-\delta}) \langle y \rangle^{-1} y.$$

Let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi(t) = 1$ if and only if $|t| \leq 1$ and $\text{supp} \chi \subset [-2; 2]$. Let $\tilde{\chi} = 1 - \chi$. For $R \geq 1$ and $t \in \mathbb{R}$, we set $\chi_R(t) = \chi(t/R)$ and $\tilde{\chi}_R(t) = \tilde{\chi}(t/R)$. We also set $g_{\delta,R}(y) = \tilde{\chi}_R(\langle y \rangle)^2 g_\delta(y)$. Recall that we set $\beta_{lr} = \min(\rho_{lr}, \beta)$.

First, we show a kind of weighted Mourre estimate at infinity for the position operators Q (meaning for large $|Q|$), which can be seen as an energy localised (i.e. localised in H) Putnam positivity, that is also localised in $|Q|$ at infinity. It should be compared with Section 2 in [La1].

Proposition 8.1. *Assume Assumption 1.1. Under Assumption 3.2, take any compact interval $\mathcal{I}' \subset \mathring{\mathcal{J}}$, the interior of \mathcal{J} . Let δ be a small enough positive number (depending only on the potential) and $s = (1 + \delta)/2$. There exist $c_1 > 0$ and $R_1 > 1$ such that, for $R \geq R_1$, there exists a bounded, self-adjoint operator B_R such that, for $f \in L^2(\mathbb{R}^d)$ with $E_{\mathcal{I}'}(H)f = f$, we have the estimate:*

$$(8.2) \quad \begin{aligned} \langle f, [H, iB_R]f \rangle &\geq c_1 \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|^2 - O(R^{-\gamma}) \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\| \\ &\quad - O(R^{-\gamma-1}), \end{aligned}$$

with $\gamma = \beta - \delta > 0$ when $\alpha = 1$ and $\gamma = 1$ when $\alpha > 1$. Here $B_R = g_{\delta,R}(Q) \cdot P + P \cdot g_{\delta,R}(Q)$. The “ O ” terms in the estimate can be chosen independent of f when f stays in a bounded set for the norm $\|\langle Q \rangle^{-s} \cdot\|$.

Remark 8.2. In fact, we can give the precise size that δ should have in Proposition 8.1. We demand that $\delta < \min(\beta, \rho_{sr}, \rho_{lr}, \rho'_{sr})$ and also $\delta < \beta + \beta_{lr} - 1$. Recall that $\beta + \beta_{lr} > 1$, by Assumption 3.2.

Denoting by c the infimum of \mathcal{J} , one can take $c_1 = \delta c/2$ in (8.2).

Proof. We choose δ according to Remark 8.2. We take f satisfying $E_{\mathcal{I}'}(H)f = f$ and belonging to some fix bounded set for the norm $\|\langle Q \rangle^{-s} \cdot\|$. Let $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{R})$ such that $\theta = 1$ on \mathcal{I}' and $\text{supp } \theta \subset \mathring{\mathcal{J}}$. We have $\theta(H)f = f$. Take R_1 large enough such that, for $R \geq R_1$, $\tilde{\chi}_R V_c = 0$. In particular, setting $G_{\delta,R} = iP \cdot g_{\delta,R}$,

$$\begin{aligned} \langle f, [V_c, iB_R]f \rangle &= \langle V_c f, g_{\delta,R}(Q) \cdot iP f \rangle + \langle g_{\delta,R}(Q) \cdot iP f, V_c f \rangle \\ &\quad + \langle f, G_{\delta,R}(Q) V_c f \rangle \\ &= 0 \end{aligned}$$

and $\langle f, [V_{lr}, iB_R]f \rangle = -\langle f, g_{\delta,R}(Q) \cdot (\nabla V_{lr})f \rangle$. The other contributions of the potential are given by

$$\begin{aligned} \langle f, [V_{sr}, iB_R]f \rangle &= \langle V_{sr} g_{\delta,R}(Q) f, iP f \rangle + \langle iP f, V_{sr} g_{\delta,R}(Q) f \rangle \\ &\quad + \langle f, G_{\delta,R}(Q) V_{sr} f \rangle, \\ \langle f, [W_{\alpha\beta}, iB_R]f \rangle &= \langle W_{\alpha\beta} g_{\delta,R}(Q) f, iP f \rangle + \langle iP f, W_{\alpha\beta} g_{\delta,R}(Q) f \rangle \\ &\quad + \langle f, G_{\delta,R}(Q) W_{\alpha\beta} f \rangle. \end{aligned}$$

In particular, the terms

$$\langle f, G_{\delta,R}(Q) V_{sr} f \rangle, \quad \langle f, G_{\delta,R}(Q) W_{\alpha\beta} f \rangle, \quad \text{and} \quad \langle f, g_{\delta,R}(Q) \cdot (\nabla V_{lr})f \rangle$$

are $O(R^{-\epsilon}) \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|^2$, for some $\epsilon > 0$. We shall evaluate the size of the other terms. To this end, we shall repeatedly make use of Lemma C.5, of Lemma C.6 and of the fact that the term $\|\langle Q \rangle^{-s} f\|$ stays in a bounded region, for the considered f . Note that those lemmata follow from the regularity of H w.r.t. $\langle Q \rangle$.

Writing

$$\begin{aligned} &\langle V_{sr} g_{\delta,R}(Q) f, iP f \rangle \\ &= \langle V_{sr} g_{\delta,R}(Q) f, iP \theta(H) f \rangle \\ &= \langle \langle Q \rangle^s V_{sr} g_{\delta,R}(Q) \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f, [\tilde{\chi}_R(\langle Q \rangle), iP \theta(H)] \langle Q \rangle^s \cdot \langle Q \rangle^{-s} f \rangle \\ &\quad + \langle \langle Q \rangle^{2s} V_{sr} g_{\delta,R}(Q) \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f, \langle Q \rangle^{-s} iP \theta(H) \langle Q \rangle^s \cdot \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f \rangle, \end{aligned}$$

the first term is $O(R^{\delta-1-\rho_{sr}}) \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|$ and the second term is at most of size $O(R^{\delta-\rho_{sr}}) \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|^2$, by Lemma C.6. Notice that $O(R^{\delta-1-\rho_{sr}}) = O(R^{-\gamma})$ and that $\delta - \rho_{sr} < 0$.

To evaluate the contribution of $W_{\alpha\beta}$, we only look at the term $\langle W_{\alpha\beta} g_{\delta,R}(Q) f, \nabla f \rangle$, since the other terms can be treated in a similar way. We write

$$\begin{aligned} &\langle W_{\alpha\beta} g_{\delta,R}(Q) f, iP f \rangle \\ &= \langle \tilde{\chi}_R(\langle Q \rangle)^2 W_{\alpha\beta} g_{\delta,R}(Q) \theta(H) f, iP \theta(H) f \rangle \\ &= \langle \tilde{\chi}_{R/2}(\langle Q \rangle) W_{\alpha\beta} g_{\delta,R}(Q) \theta(H) \tilde{\chi}_R(\langle Q \rangle) f, iP \theta(H) \tilde{\chi}_R(\langle Q \rangle) f \rangle \\ &\quad + \langle \tilde{\chi}_{R/2}(\langle Q \rangle) W_{\alpha\beta} g_{\delta,R}(Q) \theta(H) \tilde{\chi}_R(\langle Q \rangle) f, [\tilde{\chi}_R(\langle Q \rangle), iP \theta(H)] f \rangle \\ &\quad + \langle \tilde{\chi}_{R/2}(\langle Q \rangle) W_{\alpha\beta} g_{\delta,R}(Q) [\tilde{\chi}_R(\langle Q \rangle), \theta(H)] f, iP \theta(H) \tilde{\chi}_R(\langle Q \rangle) f \rangle \\ &\quad + \langle \tilde{\chi}_{R/2}(\langle Q \rangle) W_{\alpha\beta} g_{\delta,R}(Q) [\tilde{\chi}_R(\langle Q \rangle), \theta(H)] f, [\tilde{\chi}_R(\langle Q \rangle), iP \theta(H)] f \rangle. \end{aligned}$$

The second and third terms are $O(R^{\delta-\beta}) \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|$ and the last term is $O(R^{\delta-1-\beta})$, by Lemma C.6.

We now focus on the first term. We write

$$\begin{aligned}
(8.3) \quad & \langle \tilde{\chi}_{R/2}(\langle Q \rangle) W_{\alpha\beta} g_\delta(Q) \theta(H) \tilde{\chi}_R(\langle Q \rangle) f, i P \theta(H) \tilde{\chi}_R(\langle Q \rangle) f \rangle \\
&= \langle \tilde{\chi}_{R/2}(\langle Q \rangle) W_{\alpha\beta} g_\delta(Q) \theta(H_0) \tilde{\chi}_R(\langle Q \rangle) f, i P \theta(H_0) \tilde{\chi}_R(\langle Q \rangle) f \rangle \\
&+ \langle \tilde{\chi}_{R/2}(\langle Q \rangle) W_{\alpha\beta} g_\delta(Q) (\theta(H) - \theta(H_0)) \tilde{\chi}_R(\langle Q \rangle) f, i P \theta(H_0) \tilde{\chi}_R(\langle Q \rangle) f \rangle \\
&+ \langle \tilde{\chi}_{R/2}(\langle Q \rangle) W_{\alpha\beta} g_\delta(Q) \theta(H_0) \tilde{\chi}_R(\langle Q \rangle) f, i P (\theta(H) - \theta(H_0)) \tilde{\chi}_R(\langle Q \rangle) f \rangle \\
&+ \langle \tilde{\chi}_{R/2}(\langle Q \rangle) W_{\alpha\beta} g_\delta(Q) (\theta(H) - \theta(H_0)) \tilde{\chi}_R(\langle Q \rangle) f, \\
&\quad i P (\theta(H) - \theta(H_0)) \tilde{\chi}_R(\langle Q \rangle) f \rangle.
\end{aligned}$$

By Lemma C.5, the second and third terms on the r.h.s. of (8.3) are at most of size $O(R^{\delta+1-\beta-\beta_{lr}}) \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|^2$, whereas the fourth one is seen to be $O(R^{\delta+1-\beta-2\beta_{lr}}) \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|^2$. We write the first one as

$$\begin{aligned}
& \langle \tilde{\chi}_{R/2}(\langle Q \rangle) W_{\alpha\beta} g_\delta(Q) \theta(H_0) \tilde{\chi}_R(\langle Q \rangle) f, i P \theta(H_0) \tilde{\chi}_R(\langle Q \rangle) f \rangle \\
&= \langle W_{\alpha\beta} [\tilde{\chi}_{R/2}(\langle Q \rangle) g_\delta(Q), \theta(H_0)] \tilde{\chi}_R(\langle Q \rangle) f, i P \theta(H_0) \tilde{\chi}_R(\langle Q \rangle) f \rangle \\
&\quad + \langle W_{\alpha\beta} \theta(H_0) g_\delta(Q) \tilde{\chi}_R(\langle Q \rangle) f, i P \theta(H_0) \tilde{\chi}_R(\langle Q \rangle) f \rangle.
\end{aligned}$$

By the above arguments, the first term on the r.h.s is $O(R^{\delta-\beta}) \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|^2$. So is also the last term by Proposition 2.1.

We are left with the contribution of H_0 in the l.h.s. of (8.2). A direct computation gives $[H_0, iB_R] = P^T \cdot \mathcal{G}_{\delta,R} \cdot P - h_{\delta,R}$ where the entries of the $d \times d$ -matrix valued function $\mathcal{G}_{\delta,R}$ on \mathbb{R}^d are given by

$$\partial_k (\tilde{\chi}_R(\langle \cdot \rangle)^2 (g_\delta)_j)(y).$$

Here $g_\delta(y) = ((g_\delta)_1(y), \dots, (g_\delta)_d(y))^T$ and T denotes the transposition. The real valued function $h_{\delta,R}$ on \mathbb{R}^d is given by

$$h_{\delta,R}(y) = \sum_{1 \leq j, k \leq d} \partial_{kkj}^3 (\tilde{\chi}_R(\langle \cdot \rangle)^2 (g_\delta)_j)(y).$$

The contribution of $h_{\delta,R}$ to (8.2) is seen to be $O(R^{\delta-2}) = O(R^{-\gamma-1})$. Since

$$\partial_k (\tilde{\chi}_R(\langle \cdot \rangle)^2 (g_\delta)_j)(y) = \tilde{\chi}_R(\langle y \rangle)^2 \partial_k ((g_\delta)_j)(y) + 2(2 - \langle y \rangle^{-\delta}) \tilde{\chi}_R(\langle y \rangle) \tilde{\chi}'_R(\langle y \rangle) \frac{y_j y_k}{\langle y \rangle^2},$$

$2(2 - \langle \cdot \rangle^{-\delta}) \tilde{\chi}_R(\langle \cdot \rangle) \tilde{\chi}'_R(\langle \cdot \rangle) \geq 0$, and the matrix $(y_j y_k \langle y \rangle^{-2})_{1 \leq j, k \leq d}$ is nonnegative,

$$\langle f, P^T \cdot \mathcal{G}_{\delta,R} \cdot P f \rangle \geq \langle f, P^T \cdot \tilde{\chi}_R(\langle Q \rangle)^2 \mathcal{G}_\delta(Q) \cdot P f \rangle$$

where the entries of the $d \times d$ -matrix valued function \mathcal{G}_δ on \mathbb{R}^d are given by $\partial_k ((g_\delta)_j)(y)$. For $y \in \mathbb{R}^d$, $\mathcal{G}_\delta(y)$ is the sum of two nonnegative matrices, namely

$$\begin{aligned}
\mathcal{G}_\delta(y) &= \frac{(2 - \langle y \rangle^{-\delta})}{\langle y \rangle} \left(\delta_{jk} - \frac{y_j y_k}{\langle y \rangle^2} \right)_{1 \leq j, k \leq d} + \frac{\delta}{\langle y \rangle^{1+\delta}} \left(\frac{y_j y_k}{\langle y \rangle^2} \right)_{1 \leq j, k \leq d} \\
&\geq \frac{\delta}{\langle y \rangle^{1+\delta}} \left(\delta_{jk} - \frac{y_j y_k}{\langle y \rangle^2} \right)_{1 \leq j, k \leq d} + \frac{\delta}{\langle y \rangle^{1+\delta}} \left(\frac{y_j y_k}{\langle y \rangle^2} \right)_{1 \leq j, k \leq d} \\
&\geq \frac{\delta}{\langle y \rangle^{1+\delta}} I_d,
\end{aligned}$$

where I_d is the $d \times d$ identity matrix. This yields

$$\langle f, P^T \cdot \mathcal{G}_{\delta,R} \cdot P f \rangle \geq \delta \langle f, P^T \cdot \tilde{\chi}_R(\langle Q \rangle)^2 \langle Q \rangle^{-2s} P f \rangle.$$

We write

$$\begin{aligned}
& \langle f, P^T \cdot \tilde{\chi}_R(\langle Q \rangle)^2 \langle Q \rangle^{-2s} P f \rangle \\
&= \langle \theta(H) f, P^T \cdot \tilde{\chi}_R(\langle Q \rangle)^2 \langle Q \rangle^{-2s} P \theta(H) f \rangle \\
&= \langle f, [\theta(H) P^T, \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s}] \cdot [\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s}, P \theta(H)] f \rangle \\
&\quad + \langle f, [\theta(H) P^T, \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s}] \cdot P \theta(H) \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f \rangle \\
&\quad + \langle \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f, \theta(H) P^T \cdot [\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s}, P \theta(H)] f \rangle \\
&\quad + \langle \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f, \theta(H) H_0 \theta(H) \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f \rangle.
\end{aligned}$$

By Lemma C.6, the first term is $O(R^{-2}) = O(R^{-\gamma-1})$, the second and third ones are $O(R^{-1}) \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|$, thus also $O(R^{-\gamma}) \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|$. Writing $H_0 = H - V$ in the last term and using the fact that $\theta(H) V \langle Q \rangle^{\beta_{lr}}$ is bounded, this last term is

$$\geq c \|\theta(H) \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|^2 - O(R^{-\beta_{lr}}) \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|^2,$$

where c is the infimum of \mathcal{J} . Now, we write

$$\begin{aligned}
& \|\theta(H) \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|^2 \\
&= \langle [\theta(H), \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s}] f, [\theta(H), \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s}] f \rangle \\
&\quad + \langle [\theta(H), \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s}] f, \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f \rangle \\
&\quad + \langle \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f, [\theta(H), \tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s}] f \rangle \\
&\quad + \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|^2.
\end{aligned}$$

By Lemma C.6 again, the first term is $O(R^{-2}) = O(R^{-\gamma-1})$, the second and third ones are $O(R^{-1}) \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\| = O(R^{-\gamma}) \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f\|$. Gathering all the previous estimates and taking R_1 large enough, we get (8.2) with $c_1 = \delta c/2$. \square

Now we are in position to prove our first main result, namely Theorem 1.5. To this end, we use the characterization of the LAP in terms of so called “special sequences”, that was introduced in [GJ1].

Proof of Theorem 1.5. Without loss of generality, the length of \mathcal{I} may be assumed small enough. In particular, we can find a compact interval \mathcal{J} satisfying Assumption 3.2 such that $\mathcal{I} \subset \mathcal{J}$. Since the validity of (1.2) for some $s > 1/2$ implies the validity of (1.2) for any $s' \geq s$, we may choose $s > 1/2$ as close to $1/2$ as we want. Let $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{R})$ such that $\theta = 1$ on \mathcal{I} and $\text{supp } \theta \subset \mathcal{J}$. By Proposition 3.7, (3.3) holds true. Multiplying each term on both sides by $\chi(H) \Pi^\perp$, with $\chi \theta = \chi$, and shrinking the size of the support of χ so much that $\|K \chi(H) \Pi^\perp\| \leq c$, we get (3.3) with $2c$ replaced by c , $\theta(H)$ replaced by $\chi(H) \Pi^\perp$, and $K = 0$. This can be done with the requirement that $\chi = 1$ on a small compact interval. Therefore we may assume that, for \mathcal{I} , \mathcal{J} , and θ , as above, we have the following strict, projected Mourre estimate

$$(8.4) \quad \Pi^\perp \theta(H) [H, iA] \theta(H) \Pi^\perp \geq c \theta(H)^2 \Pi^\perp.$$

Recall that $\theta(H) \in \mathcal{C}^\infty(\langle Q \rangle)$ and $\theta(H) \Pi \in \mathcal{C}^\infty(\langle Q \rangle)$, by Lemma 3.1 and by Corollary 5.3, respectively. Thus $\theta(H) \Pi^\perp \in \mathcal{C}^\infty(\langle Q \rangle)$ and we can apply Proposition 3.2 in [GJ2]. Therefore the LAP (1.2) is equivalent to the following statement:

Take a sequence $(f_n, z_n)_{n \in \mathbb{N}}$ such that, for all n , $z_n \in \mathbb{C}$, $\Re z_n \in \mathcal{I}$, $\Im z_n \neq 0$, $f_n \in \mathcal{D}(H)$, $\Pi^\perp f_n = f_n$, $\theta(H) f_n = f_n$, and $(H - z_n) f_n \in \mathcal{D}(\langle Q \rangle^s)$. Assume further

that $\Im z_n \rightarrow 0$, $\|\langle Q \rangle^s (H - z_n) f_n\| \rightarrow 0$, and that $(\|\langle Q \rangle^{-s} f_n\|)_{n \in \mathbb{N}}$ converges to some real number η . Then $\eta = 0$.

We shall prove this statement. Let us consider such a sequence $(f_n, z_n)_{n \in \mathbb{N}}$. Take $R \geq 1$. Notice that $\chi_R(\langle Q \rangle) f_n$ actually belongs to $\mathcal{D}(H) \cap \mathcal{D}(\langle Q \rangle)$. Note also that the operator $A^\perp := \Pi^\perp \theta(H) A \theta(H) \Pi^\perp$ is well-defined on $\mathcal{D}(\langle Q \rangle)$, since $P\theta(H)$ is bounded and preserves, together with $\theta(H)$ and Π^\perp , the set $\mathcal{D}(\langle Q \rangle)$. Since H commutes with $\theta(H) \Pi^\perp$, we derive from (8.4) applied to $\chi_R(\langle Q \rangle) f_n$ that

$$(8.5) \quad \langle \chi_R(\langle Q \rangle) f_n, [H, iA^\perp] \chi_R(\langle Q \rangle) f_n \rangle \geq c \|\theta(H) \Pi^\perp \chi_R(\langle Q \rangle) f_n\|^2.$$

Since $\theta(H) \Pi^\perp$ is smooth w.r.t. $\langle Q \rangle$,

$$\begin{aligned} \theta(H) \Pi^\perp \chi_R(\langle Q \rangle) f_n &= \chi_R(\langle Q \rangle) f_n + [\theta(H) \Pi^\perp, \chi_R(\langle Q \rangle)] \langle Q \rangle^s \cdot \langle Q \rangle^{-s} f_n \\ &= \chi_R(\langle Q \rangle) f_n + O(R^{s-1}), \end{aligned}$$

thanks to Lemma C.6. The above $O(R^{s-1})$ and the following "O" are all independent of n . Inserting this information in (8.5), we get

$$(8.6) \quad \begin{aligned} \langle \chi_R(\langle Q \rangle) f_n, [H, iA^\perp] \chi_R(\langle Q \rangle) f_n \rangle &\geq c \|\chi_R(\langle Q \rangle) f_n\|^2 \\ &\quad + O(R^{s-1}) \|\chi_R(\langle Q \rangle) f_n\| + O(R^{2s-2}). \end{aligned}$$

Now, we need information on the f_n for "large $\langle Q \rangle$ ". Let \mathcal{I}' a compact interval such that $\text{supp } \theta \subset \mathcal{I}' \subset \mathring{\mathcal{J}}$. Since $f_n = \theta(H) f_n$ and $E_{\mathcal{I}'} \theta = \theta$, $E_{\mathcal{I}'} f_n = E_{\mathcal{I}'}(H) \theta(H) f_n = \theta(H) f_n = f_n$. Furthermore, the sequence $(\|\langle Q \rangle^{-s} f_n\|)_n$ is bounded since it converges to η , by assumption. Therefore we can apply Proposition 8.1 to $f = f_n$ (choosing s close enough to $1/2$, requiring in particular that $s < \gamma$), yielding (8.2) with f replaced by f_n and with n -independent "O's". As in [GJ1] (cf. Corollary 3.2), we deduce from this that, for $R \geq R_1$,

$$(8.7) \quad \limsup_n \|\tilde{\chi}_R(\langle Q \rangle) \langle Q \rangle^{-s} f_n\| = O(R^{-\gamma}) + O(R^{-(\gamma+1)/2}) = O(R^{-\gamma}).$$

We rewrite the l.h.s of (8.6) as

$$\begin{aligned} &\langle \chi_R(\langle Q \rangle) f_n, [H, iA^\perp] \chi_R(\langle Q \rangle) f_n \rangle \\ &= \langle f_n, [H, i\chi_R(\langle Q \rangle) A^\perp \chi_R(\langle Q \rangle)] f_n \rangle + 2\Re \langle [H, \chi_R(\langle Q \rangle)] f_n, iA^\perp \chi_R(\langle Q \rangle) f_n \rangle. \end{aligned}$$

Since, as form,

$$[H, \chi_R(\langle Q \rangle)] = [H_0, \chi_R(\langle Q \rangle)]_\circ = -2\nabla \cdot \left(\nabla(\chi_R(\langle \cdot \rangle)) \right)(Q) + \left(\Delta(\chi_R(\langle \cdot \rangle)) \right)(Q),$$

and since $\langle Q \rangle^{-1} \nabla A^\perp$ is bounded, we obtain, using (8.7),

$$2\Re \langle [H, \chi_R(\langle Q \rangle)] f_n, iA^\perp \chi_R(\langle Q \rangle) f_n \rangle = O(R^{s-\gamma}) \|\chi_R(\langle Q \rangle) f_n\|.$$

Therefore (8.6) yields

$$(8.8) \quad \begin{aligned} \langle f_n, [H, i\chi_R(\langle Q \rangle) A^\perp \chi_R(\langle Q \rangle)] f_n \rangle &\geq c \|\chi_R(\langle Q \rangle) f_n\|^2 \\ &\quad + O(R^{s-\gamma}) \|\chi_R(\langle Q \rangle) f_n\| + O(R^{2s-2}). \end{aligned}$$

Expanding the commutator as in [GJ1] (cf. Proposition 2.15), we see that

$$(8.9) \quad \lim_n \langle f_n, [H, i\chi_R(\langle Q \rangle) A^\perp \chi_R(\langle Q \rangle)] f_n \rangle = 0.$$

Using (8.9) in (8.8), we deduce that

$$\limsup_n \|\chi_R(\langle Q \rangle) f_n\| = O(R^{s-\gamma}),$$

with $s - \gamma < 0$. It follows from this and (8.7) that $\eta = 0$. \square

9. SYMBOL-LIKE LONG RANGE POTENTIALS.

This section is devoted to the

Proof of Theorem 1.15. Let H_1 be the self-adjoint operator $H_0 + V_{lr}(Q)$ on $\mathcal{D}(H_0)$. Thanks to the assumption on V_{lr} , H_1 is actually the Weyl quantization p^w of the symbol $p \in S(\langle \xi \rangle^2, g)$ defined by $p(x; \xi) = |\xi|^2 + V_{lr}(x)$ (see Appendix A for details). Now we redo the proofs of Theorems 1.5 and 1.11, replacing H_0 by H_1 at some appropriate places. More precisely, we perform this replacement exactly when the original proofs use the “decay” in $\langle Q \rangle$ of $\theta(H) - \theta(H_0)$.

First, we claim that the last statement in Proposition 2.1 is valid if H_0 is replaced by H_1 . We then follow the proof of Lemma 4.3 in [GJ2] and arrive at (4.7) with $\theta(|\xi|^2)\theta(|\xi \mp k\hat{x}|^2)$ replaced by $\theta(|\xi|^2 + V_{lr}(x))\theta(|\xi \mp k\hat{x}|^2 + V_{lr}(x))$. Since the latter also vanishes for small enough support of θ , we conclude as in [GJ2].

For any $\ell \geq 0$ and any $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{C})$, $\langle P \rangle^\ell \theta(H_1)$ is bounded by pseudodifferential calculus (cf. Appendix A). Therefore, the last statement in Proposition 2.4 holds true with H_0 is replaced by H_1 .

We can check that the result in Lemma 3.6 holds true with H_0 replaced by H_1 . Thus, performing the same replacement in (3.2), we get the result of Proposition 3.4. We derive the Mourre estimate of Proposition 3.7 by the same proof. Also with the same proofs, we get the results of Proposition 4.1, Corollary 4.2, Proposition 5.1, Corollary 5.2, and Corollary 5.3.

In the proof of Proposition 6.1, we modify the argument leading to the bound (6.5). Again, we replace H_0 by H_1 . We notice that $H_1(F) = e^{F(Q)} H_1 e^{-F(Q)}$ is also a pseudodifferential operator in $\mathcal{C}^1(\langle Q \rangle)$ such that, for $\epsilon \in [0; 1]$, the operator $\langle Q \rangle^\epsilon (m + H_1(F))^{-1} \langle Q \rangle^{-\epsilon}$ is bounded, uniformly w.r.t. $\lambda \geq 1$. Then, we can follow the end of the proof of Proposition 6.1 with β_{lr} replaced by β , $H_0(F)$ by $H_1(F)$, and V by $V - V_{lr}$.

Next, we redo the proof of Proposition 7.2 without change. In the proof of Proposition 8.1, we only change the treatment of (8.3) in the following way. We can check that the results in Lemma C.5 are valid with H_0 replaced by H_1 and β_{lr} by β . Concerning the first term on the r.h.s of (8.3), we only need to point out that $\langle P \rangle^\ell \theta(H_1)$ is bounded for any ℓ , by pseudodifferential calculus. We thus obtain the result of Proposition 8.1. Finally, we recover the result of Theorem 1.5 by the same proof. \square

APPENDIX A. STANDART PSEUDODIFFERENTIAL CALCULUS.

In this appendix, we briefly review some basic facts about pseudodifferential calculus. We refer to [Hö][Chapters 18.1, 18.4, 18.5, and 18.6] for a traditional study of the subject but also to [Be, Bo1, Bo2, BC, Le] for a modern and powerful version.

Denote by $\mathcal{S}(M)$ the Schwartz space on the space M and by \mathcal{F} the Fourier transform on \mathbb{R}^d given by

$$\mathcal{F}u(\xi) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} u(x) dx,$$

for $\xi \in \mathbb{R}^d$ and $u \in \mathcal{S}(\mathbb{R}^d)$. For test functions $u, v \in \mathcal{S}(\mathbb{R}^d)$, let $\Omega(u, v)$ and $\Omega'(u, v)$ be the functions in $\mathcal{S}(\mathbb{R}^{2d})$ defined by

$$\begin{aligned}\Omega(u, v)(x, \xi) &:= \bar{v}(x) \mathcal{F}u(\xi) e^{ix \cdot \xi}, \\ \Omega'(u, v)(x, \xi) &:= (2\pi)^{-d} \int_{\mathbb{R}^d} u(x - y/2) \bar{v}(x + y/2) e^{-iy \cdot \xi} dy,\end{aligned}$$

respectively. Given a distribution $b \in \mathcal{S}'(T^*\mathbb{R}^d)$, the formal quantities

$$\begin{aligned}(2\pi)^{-d} \int_{\mathbb{R}^{3d}} e^{i(x-y) \cdot \xi} b(x, \xi) v(x) u(y) dx dy d\xi, \\ (2\pi)^{-d} \int_{\mathbb{R}^{3d}} e^{i(x-y) \cdot \xi} b((x+y)/2, \xi) u(x) u(y) dx dy d\xi\end{aligned}$$

are defined by the duality brackets $\langle b, \Omega(u, v) \rangle$ and $\langle b, \Omega'(u, v) \rangle$, respectively. They define continuous operators from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ that we denote by $\text{Op } b(x, D_x)$ and $b^w(x, D_x)$ respectively. Sometimes we simply write $\text{Op } b$ and b^w , respectively. Choosing on the phase space $T^*\mathbb{R}^d$ a metric g and a weight function m with appropriate properties (cf., admissible metric and weight in [Le]), let $S(m, g)$ be the space of smooth functions on $T^*\mathbb{R}^d$ such that, for all $k \in \mathbb{N}$, there exists $c_k > 0$ so that, for all $x^* = (x, \xi) \in T^*\mathbb{R}^d$, all $(t_1, \dots, t_k) \in (T^*\mathbb{R}^d)^k$,

$$(A.1) \quad |a^{(k)}(x^*) \cdot (t_1, \dots, t_k)| \leq c_k m(x^*) g_{x^*}(t_1)^{1/2} \dots g_{x^*}(t_k)^{1/2}.$$

Here, $a^{(k)}$ denotes the k -th derivative of the function a . We equip the vector space $S(m, g)$ with the semi-norms $\|\cdot\|_{\ell, S(m, g)}$ defined by $\max_{0 \leq k \leq \ell} c_k$, where the c_k are the best constants in (A.1). $S(m, g)$ is a Fréchet space. The space of operators $\text{Op } b(x, D_x)$ (resp. $b^w(x, D_x)$) when $b \in S(m, g)$ has nice properties (cf., [Hö, Le]). Defining $x^* = (x, \xi) \in T^*\mathbb{R}^d$, we stick here to the following metrics

$$(A.2) \quad (g_0)_{x^*} := \frac{dx^2}{\langle x \rangle^2} + \frac{d\xi^2}{\langle \xi \rangle^2} \quad \text{and} \quad (g_\alpha)_{x^*} := \frac{dx^2}{\langle x \rangle^{2(1-\alpha)}} + \frac{d\xi^2}{\langle \xi \rangle^2},$$

for $0 < \alpha < 1$, and to weights of the form, for $p, q \in \mathbb{R}$,

$$(A.3) \quad m(x^*) := \langle x \rangle^p \langle \xi \rangle^q.$$

The gain of the calculus associated to each metric in (A.2) is given respectively by

$$(A.4) \quad h_0(x^*) := \langle x \rangle^{-1} \langle \xi \rangle^{-1} \quad \text{and} \quad h_\alpha(x^*) = \langle x \rangle^{1-\alpha} \langle \xi \rangle^{-1}.$$

Take weights m_1, m_2 as in (A.3), let g be g_0 or g_α , and denote by h the gain of \tilde{g} . For any $a \in S(m_1, g)$ and $b \in S(m_2, g)$, there are a symbol $a \#_r b \in S(m_1 m_2, g)$ and a symbol $a \# b \in S(m_1 m_2, g)$ such that $\text{Op } a \text{Op } b = \text{Op } (a \#_r b)$ and $a^w b^w = (a \# b)^w$. The maps $(a, b) \mapsto a \#_r b$ and $(a, b) \mapsto a \# b$ are continuous and so are also $(a, b) \mapsto a \#_r b - ab \in S(m_1 m_2 h, g)$ and $(a, b) \mapsto a \# b - ab \in S(m_1 m_2 h, g)$. If $a \in S(m_1, g)$, there exists $c \in S(m_1, g)$ such that $a^w = \text{Op } c$. The maps $a \mapsto c$ and $a \mapsto c - a \in S(m_1 m_2 h, g)$ are continuous. If $a \in S(\langle \xi \rangle^m, g)$ for $m \in \mathbb{N}$, a^w and $\text{Op } a$ are bounded from $\mathcal{H}^m(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ and the corresponding operator norms are controlled above by some appropriate semi-norm of a in $S(\langle \xi \rangle^m, g)$. In particular, they are bounded on $L^2(\mathbb{R}^d)$, if $a \in S(1, g)$. For $a \in S(1, g)$,

$$(A.5) \quad \text{Op } a \text{ is compact} \iff a^w \text{ is compact} \iff \lim_{|x^*| \rightarrow \infty} a(x^*) = 0.$$

Finally, we recall the following result on some smooth functional calculus for pseudodifferential operators associated to some admissible metric g . This result is essentially contained in [Bo1] (see [GJ2, Le], for details). We also use it for $g = g_0$ or $g = g_\alpha$.

For $\rho \in \mathbb{R}$, we denote by \mathcal{S}^ρ the set of smooth functions $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ such that, for all $k \in \mathbb{N}$, $\sup_{t \in \mathbb{R}} \langle t \rangle^{k-\rho} |\partial_t^k \varphi(t)| < \infty$. If we take a real symbol $a \in S(m, g)$, then the operator a^w is self-adjoint on the domain $\mathcal{D}(a^w) = \{u \in L^2(\mathbb{R}_x^d); a^w u \in L^2(\mathbb{R}_x^d)\}$. In particular, the operator $\varphi(a^w)$ is well defined by the functional calculus if φ is a borelean function on \mathbb{R} . We assume that $m \geq 1$. A real symbol $a \in S(m, g)$ is said elliptic if $(i - a)^{-1}$ belongs to $S(m^{-1}, g)$. Recall that h denotes the gain of the symbolic calculus in $S(m, g)$.

Theorem A.1. *Let $m \geq 1$ and $a \in S(m, g)$ be real and elliptic. Take a function $\varphi \in \mathcal{S}^\rho$. Then $\varphi(a) \in S(m^\rho, g)$ and there exists $b \in S(hm^\rho, g)$ such that*

$$(A.6) \quad \varphi(a^w(x, D)) = (\varphi(a))^w(x, D) + b^w(x, D).$$

APPENDIX B. REGULARITY W.R.T. AN OPERATOR.

For sake of completeness, we recall here important facts on the regularity w.r.t. to a self-adjoint operator. Further details can be found in [ABG, DG, GJ2, GGM, GGé].

Let \mathcal{H} be a complex Hilbert space. The scalar product $\langle \cdot, \cdot \rangle$ in \mathcal{H} is right linear and $\|\cdot\|$ denotes the corresponding norm and also the norm in $\mathcal{B}(\mathcal{H})$, the space of bounded operators on \mathcal{H} . Let M be a self-adjoint operator in \mathcal{H} . Let T be a closed operator in \mathcal{H} . The form $[T, M]$ is defined on $(\mathcal{D}(M) \cap \mathcal{D}(T)) \times (\mathcal{D}(M) \cap \mathcal{D}(T))$ by

$$(B.1) \quad \langle f, [T, M]g \rangle := \langle T^* f, Mg \rangle - \langle Mf, Tg \rangle.$$

If T is a bounded operator on \mathcal{H} and $k \in \mathbb{N}$, we say that $T \in \mathcal{C}^k(M)$ if, for all $f \in \mathcal{H}$, the map $\mathbb{R} \ni t \mapsto e^{itM} T e^{-itM} f \in \mathcal{H}$ has the usual \mathcal{C}^k regularity. The following characterization is available.

Proposition B.1. [ABG, p. 250]. *Let $T \in \mathcal{B}(\mathcal{H})$. Are equivalent:*

- (1) $T \in \mathcal{C}^1(M)$.
- (2) *The form $[T, M]$ defined on $\mathcal{D}(M) \times \mathcal{D}(M)$ extends to a bounded form on $\mathcal{H} \times \mathcal{H}$ associated to a bounded operator denoted by $\text{ad}_M^1(T) := [T, M]_\circ$.*
- (3) *T preserves $\mathcal{D}(M)$ and the operator $TM - MT$, defined on $\mathcal{D}(M)$, extends to a bounded operator on \mathcal{H} .*

It immediately follows that $T \in \mathcal{C}^k(M)$ if and only if the iterated commutators $\text{ad}_M^p(T) := [\text{ad}_M^{p-1}(T), M]_\circ$ are bounded for $p \leq k$.

It turns out that $T \in \mathcal{C}^k(M)$ if and only if, for a z outside $\sigma(T)$, the spectrum of T , $(T - z)^{-1} \in \mathcal{C}^k(M)$. Now, let N be a self-adjoint operator in \mathcal{H} . It is natural to say that $N \in \mathcal{C}^k(M)$ if $(N - z)^{-1} \in \mathcal{C}^k(M)$ for some $z \notin \sigma(N)$. In that case, $(N - z)^{-1} \in \mathcal{C}^k(M)$, for all $z \notin \mathbb{R}$. Lemma 6.2.9 and Theorem 6.2.10 in [ABG] gives the following characterization of this regularity:

Theorem B.2. [ABG]. *Let M and N be two self-adjoint operators in the Hilbert space \mathcal{H} . For $z \notin \sigma(N)$, set $R(z) := (N - z)^{-1}$. The following points are equivalent:*

- (1) $N \in \mathcal{C}^1(M)$.

- (2) For one (then for all) $z \notin \sigma(N)$, there is a finite c such that
- (B.2) $|\langle Mf, R(z)f \rangle - \langle R(\bar{z})f, Mf \rangle| \leq c\|f\|^2$, for all $f \in \mathcal{D}(M)$.
- (3) a. There is a finite c such that for all $f \in \mathcal{D}(M) \cap \mathcal{D}(N)$:
- (B.3) $|\langle Mf, Nf \rangle - \langle Nf, Mf \rangle| \leq c(\|Nf\|^2 + \|f\|^2)$.
- b. The set $\{f \in \mathcal{D}(M); R(z)f \in \mathcal{D}(M) \text{ and } R(\bar{z})f \in \mathcal{D}(M)\}$ is a core for M , for some (then for all) $z \notin \sigma(N)$.

Note that the condition (3.b) could be uneasy to check, see [GGé]. We mention [GM][Lemma A.2] to overcome this subtlety. Note that (B.2) yields that the commutator $[M, R(z)]$ extends to a bounded operator, in the form sense. We shall denote the extension by $[M, R(z)]_\circ$. In the same way, from (B.3), the commutator $[N, M]$ extends to a unique element of $\mathcal{B}(\mathcal{D}(N), \mathcal{D}(N)^*)$ denoted by $[N, M]_\circ$. Moreover, if $N \in \mathcal{C}^1(M)$ and $z \notin \sigma(N)$,

$$(B.4) \quad [M, (N - z)^{-1}]_\circ = \underbrace{(N - z)^{-1}}_{\mathcal{H} \leftarrow \mathcal{D}(N)^*} \underbrace{[N, M]_\circ}_{\mathcal{D}(N)^* \leftarrow \mathcal{D}(N)} \underbrace{(N - z)^{-1}}_{\mathcal{D}(N) \leftarrow \mathcal{H}}.$$

Here we used the Riesz lemma to identify \mathcal{H} with its anti-dual \mathcal{H}^* . It turns out that an easier characterization is available if the domain of N is conserved under the action of the unitary group generated by M .

Theorem B.3. [ABG, p. 258]. *Let M and N be two self-adjoint operators in the Hilbert space \mathcal{H} such that $e^{itM}\mathcal{D}(N) \subset \mathcal{D}(N)$, for all $t \in \mathbb{R}$. Then $N \in \mathcal{C}^1(M)$ if and only if (B.3) holds true.*

APPENDIX C. COMMUTATOR EXPANSIONS.

In this appendix, we recall known results on functional calculus and on commutator expansions. Details can be found in [DG, GJ1, GJ2, Mø]. We then apply these results to get several facts used in the main part of the text. We make use of pseudodifferential calculus (cf. Appendix A) and of the regularity w.r.t. an operator, recalled in Appendix B.

As in Appendix A, we consider, for $\rho \in \mathbb{R}$, the set \mathcal{S}^ρ of functions $\varphi \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{C})$ such that

$$(C.1) \quad \forall k \in \mathbb{N}, \quad C_k(\varphi) := \sup_{t \in \mathbb{R}} \langle t \rangle^{-\rho+k} |\partial_t^k \varphi(t)| < \infty.$$

Equipped with the semi-norms defined by (C.1), \mathcal{S}^ρ is a Fréchet space. We recall the following result from [DG] on almost analytic extension.

Proposition C.1. [DG]. *Let $\varphi \in \mathcal{S}^\rho$ with $\rho \in \mathbb{R}$. There is a smooth function $\varphi^\mathbb{C} : \mathbb{C} \rightarrow \mathbb{C}$, called an almost analytic extension of φ , such that, for all $l \in \mathbb{N}$,*

$$(C.2) \quad \varphi^\mathbb{C}|_\mathbb{R} = \varphi, \quad |\partial_{\bar{z}} \varphi^\mathbb{C}(z)| \leq c_1 \langle \operatorname{Re}(z) \rangle^{\rho-1-l} |\operatorname{Im}(z)|^l,$$

$$(C.3) \quad \operatorname{supp} \varphi^\mathbb{C} \subset \{x + iy; |y| \leq c_2 \langle x \rangle\},$$

$$(C.4) \quad \varphi^\mathbb{C}(x + iy) = 0, \text{ if } x \notin \operatorname{supp} \varphi,$$

for constants c_1, c_2 only depending on the semi-norms (C.1) of φ in \mathcal{S}^ρ .

Next we recall Helffer-Sjöstrand's functional calculus (cf., [HeS, DG]). As in Appendix B, we consider a self-adjoint operator M acting in some complex Hilbert space \mathcal{H} . For $\rho < 0$, $k \in \mathbb{N}$, and $\varphi \in \mathcal{S}^\rho$, the bounded operators $(\partial^k \varphi)(M)$ can be recovered by

$$(C.5) \quad (\partial^k \varphi)(M) = \frac{i(k!)}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \varphi^{\mathbb{C}}(z) (z - M)^{-1-k} dz \wedge d\bar{z},$$

where the integral exists in the norm topology, by (C.2) with $l = 1$. For $\rho \geq 0$, we rely on the following approximation:

Proposition C.2. [GJ1]. *Let $\rho \geq 0$ and $\varphi \in \mathcal{S}^\rho$. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\chi = 1$ near 0 and $0 \leq \chi \leq 1$, and, for $R > 0$, let $\chi_R(t) = \chi(t/R)$. For $f \in \mathcal{D}(\langle M \rangle^\rho)$ and $k \in \mathbb{N}$, there exists*

$$(C.6) \quad (\partial^k \varphi)(M)f = \lim_{R \rightarrow +\infty} \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} (\varphi \chi_R)^{\mathbb{C}}(z) (z - M)^{-1-k} f dz \wedge d\bar{z}.$$

The r.h.s. converges for the norm in \mathcal{H} . It is independent of the choice of χ .

Notice that, for some $c > 0$ and $s \in [0; 1]$, there exists some $C > 0$ such that, for all $z = x + iy \in \{a + ib \mid 0 < |b| \leq c\langle a \rangle\}$ (like in (C.3)),

$$(C.7) \quad \|\langle M \rangle^s (M - z)^{-1}\| \leq C \langle x \rangle^s \cdot |y|^{-1}.$$

Observing that the self-adjointness assumption on B is useless, we pick from [GJ1] the following result in two parts.

Proposition C.3. [BG, DG, GJ1, Mø]. *Let $k \in \mathbb{N}^*$, $\rho < k$, $\varphi \in \mathcal{S}^\rho$, and B be a bounded operator on \mathcal{H} such that $B \in \mathcal{C}^k(M)$. As forms on $\mathcal{D}(\langle M \rangle^{k-1}) \times \mathcal{D}(\langle M \rangle^{k-1})$,*

$$(C.8) \quad [\varphi(M), B] = \sum_{j=1}^{k-1} \frac{1}{j!} (\partial^j \varphi)(M) \text{ad}_M^j(B)$$

$$(C.9) \quad + \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \varphi^{\mathbb{C}}(z) (z - M)^{-k} \text{ad}_M^k(B) (z - M)^{-1} dz \wedge d\bar{z}.$$

In particular, if $\rho \leq 1$, then $B \in \mathcal{C}^1(\varphi(M))$.

The rest of the previous expansion is estimated in

Proposition C.4. [BG, GJ1, Mø]. *Let B be a bounded operator on \mathcal{H} such that $B \in \mathcal{C}^k(M)$. Let $\varphi \in \mathcal{S}^\rho$, with $\rho < k$. Let $I_k(\varphi)$ be the rest of the development of order k (C.8) of $[\varphi(M), B]$, namely (C.9). Let $s, s' \geq 0$ such that $s' < 1$, $s < k$, and $\rho + s + s' < k$. Then, for φ staying in a bounded subset of \mathcal{S}^ρ , $\langle M \rangle^s I_k(\varphi) \langle M \rangle^{s'}$ is bounded and there exists a M and φ independent constant $C > 0$ such that $\|\langle M \rangle^s I_k(\varphi) \langle M \rangle^{s'}\| \leq C \|\text{ad}_M^k(B)\|$.*

Now, we show a serie of results needed in the main text. Most of them are more or less known. We provide proofs for completeness.

Proof of Lemma 3.1. The assumptions 1.1 and 1.2 are not required for the proof of (1). We note that $(1 + H_0)^{-1} = a^w$ and $\langle Q \rangle = b^w$, where $a(x, \xi) = (1 + |\xi|^2)^{-1}$ and $b(x, \xi) = \langle x \rangle$. Since $a \in S(\langle \xi \rangle^{-2}, g_0)$ and $b \in S(\langle x \rangle, g_0)$, where the metric g_0 defined in (A.2), the form $[(1 + H_0)^{-1}, \langle Q \rangle]$ is associated to c^w with $c \in S(h \langle \xi \rangle^{-2} \langle x \rangle, g_0) =$

$S(\langle \xi \rangle^{-3}, g_0)$, by pseudodifferential calculus. Since $S(\langle \xi \rangle^{-3}, g_0) \subset S(1, g_0)$, the form $[(1 + H_0)^{-1}, \langle Q \rangle]$ extends to bounded one on $L^2(\mathbb{R}^d)$. Similarly, we can show that the iterated commutators $\text{ad}_{\langle Q \rangle}^p((1 + H_0)^{-1})$ all extend to bounded operator on $L^2(\mathbb{R}^d)$. By the comment just after Proposition B.1, $(1 + H_0)^{-1} \in \mathcal{C}^\infty(\langle Q \rangle)$ and $H_0 \in \mathcal{C}^\infty(\langle Q \rangle)$, by definition. Since $\langle P \rangle = d^w$ with $d(x, \xi) = (1 + |\xi|^2)^{1/2}$, we can follow the same lines to prove that $\langle P \rangle^{-1} \in \mathcal{C}^\infty(\langle Q \rangle)$ and thus $\langle P \rangle \in \mathcal{C}^\infty(\langle Q \rangle)$. Similarly, $P \in \mathcal{C}^\infty(\langle Q \rangle)$. Since the form $[\langle P \rangle, \langle Q \rangle]$ is associated to bounded pseudodifferential operator, we see that $\mathcal{D}(\langle Q \rangle \langle P \rangle) = \mathcal{D}(\langle P \rangle \langle Q \rangle)$.

By a direct computation, we see that the group $e^{it\langle Q \rangle}$ (for $t \in \mathbb{R}$) preserves the Sobolev space $\mathcal{H}^2(\mathbb{R}^d)$, which is the domain of H . Furthermore the form $[H, \langle Q \rangle]$ coincide on $\mathcal{D}(H) \cap \mathcal{D}(\langle Q \rangle)$ with $[H_0, \langle Q \rangle]$. The latter is associated, by pseudodifferential calculus, to a pseudodifferential operator that is bounded from $\mathcal{H}^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. By Theorem B.3, $H \in \mathcal{C}^1(\langle Q \rangle)$ and, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$(C.10) \quad [(z - H)^{-1}, \langle Q \rangle]_\circ = (z - H)^{-1} [H, \langle Q \rangle]_\circ (z - H)^{-1}.$$

On $\mathcal{D}(\langle Q \rangle) \times \mathcal{D}(\langle Q \rangle)$, we can write the form $[(z - H)^{-1}, \langle Q \rangle]_\circ$ as

$$\begin{aligned} & [(z - H)^{-1}, \langle Q \rangle]_\circ [H, \langle Q \rangle]_\circ (z - H)^{-1} + (z - H)^{-1} [H, \langle Q \rangle]_\circ [(z - H)^{-1}, \langle Q \rangle]_\circ \\ & + (z - H)^{-1} [[H, \langle Q \rangle]_\circ, \langle Q \rangle] (z - H)^{-1}. \end{aligned}$$

Since $[[H, \langle Q \rangle]_\circ, \langle Q \rangle] = [[H_0, \langle Q \rangle]_\circ, \langle Q \rangle]$ is associated to a bounded pseudodifferential operator, $H \in \mathcal{C}^2(\langle Q \rangle)$ by Proposition B.1. Now we conclude the proof of (2) by induction, making use of (C.10) and the fact that the form $\text{ad}_{\langle Q \rangle}^p(H) = \text{ad}_{\langle Q \rangle}^p(H_0)$ extends to a bounded one, if $p \geq 2$.

Let $N = H$ or H_0 . For $z \in \mathbb{C} \setminus \mathbb{R}$, we have (C.10) with H replaced by N , thanks to (1) and (2). Using the resolvent equality for the difference $(z - N)^{-1} - (i - N)^{-1}$, we see that

$$(C.11) \quad \|[(z - N)^{-1}, \langle Q \rangle]_\circ\| \leq C \left(1 + \frac{\langle \Re z \rangle}{|\Im z|}\right).$$

where C only depends on the operator norm of $[N, \langle Q \rangle]_\circ$. Now we use (C.5) with $\varphi = \theta$ to express the form $[\theta(H), \langle Q \rangle]$ and see that it extends to a bounded one, thanks to (C.11). This shows that $\theta(N) \in \mathcal{C}^1(\langle Q \rangle)$. In a similar way, we can show by induction that $\theta(N) \in \mathcal{C}^\infty(\langle Q \rangle)$. The above arguments actually show that $P[\theta(N), \langle Q \rangle]_\circ$ is a vector of bounded operators on $L^2(\mathbb{R}^d)$. So is also $[P\theta(N), \langle Q \rangle]_\circ$ and, since $P\theta(N)$ is bounded, $P\theta(N) \in \mathcal{C}^1(\langle Q \rangle)$. Again we can derive by induction that $P\theta(N) \in \mathcal{C}^\infty(\langle Q \rangle)$.

Note that $\theta(H)\mathcal{D}(\langle Q \rangle) \subset \mathcal{D}(H) = \mathcal{D}(H_0)$. Let $z \in \mathbb{C} \setminus \mathbb{R}$. By (2), $(z - H)^{-1}$ preserves $\mathcal{D}(\langle Q \rangle)$ and, on $\mathcal{D}(\langle Q \rangle)$,

$$\langle Q \rangle (z - H)^{-1} = (z - H)^{-1} \langle Q \rangle + [\langle Q \rangle, (z - H)^{-1}]_\circ.$$

Thus $\langle Q \rangle (z - H)^{-1} \langle Q \rangle^{-1}$ is bounded and

$$\langle Q \rangle (z - H)^{-1} \langle Q \rangle^{-1} = (z - H)^{-1} + [\langle Q \rangle, (z - H)^{-1}]_\circ \langle Q \rangle^{-1}.$$

By (C.10), we see that $\langle P \rangle \langle Q \rangle (z - H)^{-1} \langle Q \rangle^{-1}$

$$= \langle P \rangle (z - H)^{-1} + \langle P \rangle (z - H)^{-1} [\langle Q \rangle, H]_\circ (z - H)^{-1} \langle Q \rangle^{-1}$$

is bounded and, for some z -independent $C' > 0$,

$$\|\langle P \rangle \langle Q \rangle (z - H)^{-1} \langle Q \rangle^{-1}\| \leq \frac{C'}{|\Im z|} \left(1 + \frac{\langle \Re z \rangle}{|\Im z|}\right).$$

Therefore, $\langle P \rangle \langle Q \rangle \theta(H) \langle Q \rangle^{-1}$ is bounded, by (C.5) with $k = 0$. This implies that $\theta(H) \mathcal{D}(\langle Q \rangle) \subset \mathcal{D}(\langle P \rangle \langle Q \rangle)$. \square

Lemma C.5. *Assume Assumptions 1.1 and 1.2. Then, for any $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{C})$ and any $\sigma \geq 0$, $\langle Q \rangle^{\beta_{lr}-\sigma}(\theta(H) - \theta(H_0)) \langle Q \rangle^\sigma$, $\langle Q \rangle^{\beta_{lr}-\sigma} P(\theta(H) - \theta(H_0)) \langle Q \rangle^\sigma$, $\langle Q \rangle^{-\sigma} P \theta(H) \langle Q \rangle^\sigma$, and $\langle Q \rangle^{-\sigma} P \theta(H_0) \langle Q \rangle^\sigma$ are bounded on $L^2(\mathbb{R}^d)$.*

Proof. We first note that, for $\delta \in [-1; 1]$, the form $[H, \langle Q \rangle^\delta] = [H_0, \langle Q \rangle^\delta]$ extends to a bounded one from $\mathcal{H}^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Thus, as in the previous proof (the one of Lemma 3.1), for $H' = H$ and $H' = H_0$, there exists $C > 0$, such that, $z \in \mathbb{C} \setminus \mathbb{R}$,

$$(C.12) \quad \|\langle P \rangle^2 \langle Q \rangle^{-\delta} (z - H')^{-1} \langle Q \rangle^\delta\| \leq \frac{C}{|\Im z|} \left(1 + \frac{\langle \Re z \rangle}{|\Im z|}\right).$$

Since, for $\delta \in [0; 1]$, we can write

$$\begin{aligned} \langle Q \rangle^{-1-\delta} (z - H')^{-1} \langle Q \rangle^{1+\delta} &= \langle Q \rangle^{-\delta} (z - H')^{-1} \langle Q \rangle^\delta \\ &\quad + \langle Q \rangle^{-1-\delta} (z - H')^{-1} [H', \langle Q \rangle]_\circ (z - H')^{-1} \langle Q \rangle^\delta \end{aligned}$$

with $[H', \langle Q \rangle]_\circ = [H_0, \langle Q \rangle]_\circ$, (C.12) implies (C.12) with δ replaced by $\delta + 1$. By induction, we get (C.12) for all $\delta \geq 0$. For $\delta \in [-1; 0]$, we can similarly show (C.12) with δ replaced by $\delta - 1$ and then, by induction, (C.12) for all $\delta \leq 0$.

For $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\begin{aligned} &\langle Q \rangle^{\beta_{lr}-\sigma} V(Q) (z - H_0)^{-1} \langle Q \rangle^\sigma \\ &= \langle Q \rangle^{\beta_{lr}} (V - V_c)(Q) \cdot \langle Q \rangle^{-\sigma} (z - H_0)^{-1} \langle Q \rangle^\sigma \\ &\quad + \langle Q \rangle^{\beta_{lr}} \chi_c(Q) \cdot V_c(Q) \langle P \rangle^{-2} \cdot \langle P \rangle^2 \langle Q \rangle^{-\sigma} (z - H_0)^{-1} \langle Q \rangle^\sigma \end{aligned}$$

By (C.12) for $H' = H$ and $\delta = \sigma - \beta_{lr}$, (C.12) for $H' = H_0$ and $\delta = \sigma$, and by the resolvent formula,

$$\langle P \rangle \langle Q \rangle^{\beta_{lr}-\sigma} \left((z - H)^{-1} - (z - H_0)^{-1} \right) \langle Q \rangle^\sigma$$

is bounded and its norm is dominated by some z -independent C' times the r.h.s. of (C.12) squared. Now, we use (C.5) with $k = 0$ to get the boundedness of $\langle P \rangle \langle Q \rangle^{\beta_{lr}-\sigma} (\theta(H) - \theta(H_0)) \langle Q \rangle^\sigma$. This shows the desired result for the two first considered operators.

The result for the last two operators follows from (C.12) and (C.5) with $k = 0$. \square

Lemma C.6. *Assume Assumptions 1.1 and 1.2 satisfied. Let $\theta \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{C})$. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}; \mathbb{R})$ with $\chi = 1$ near 0 and, for $R \geq 1$, let $\chi_R(t) = \chi(t/R)$ and $\tilde{\chi}_R(t) = 1 - \chi_R(t)$.*

(1) *For $\sigma \in [0; 1[$ and $\epsilon \geq 0$, the operators*

$$\langle Q \rangle^{\sigma-\epsilon} [\theta(H), \tilde{\chi}_R(\langle Q \rangle)]_\circ \langle Q \rangle^\sigma \quad \text{and} \quad \langle Q \rangle^{\sigma-\epsilon} [P\theta(H), \tilde{\chi}_R(\langle Q \rangle)]_\circ \langle Q \rangle^\sigma$$

are bounded on $L^2(\mathbb{R}^d)$ and their norm are $O(R^{2\sigma-1-\epsilon})$.

(2) *The operators*

$$\langle Q \rangle^{1-\beta_{lr}} [\theta(H), \chi_R(\langle Q \rangle)]_\circ \quad \text{and} \quad \langle Q \rangle^{1-\beta_{lr}} P [\theta(H), \chi_R(\langle Q \rangle)]_\circ$$

are bounded on $L^2(\mathbb{R}^d)$ and their norm are $O(R^{-\beta_{lr}})$.

Proof. We only prove (1). The proof of (2) is similar since $[\theta(H), \chi_R(\langle Q \rangle)] = -[\theta(H), \tilde{\chi}_R(\langle Q \rangle)]$ and $[P\theta(H), \chi_R(\langle Q \rangle)] = -[P\theta(H), \tilde{\chi}_R(\langle Q \rangle)]$. The form $[H, \chi_R(\langle Q \rangle)] = [H_0, \chi_R(\langle Q \rangle)]$ extends to a bounded one from $\mathcal{H}^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Furthermore,

$$[H, \tilde{\chi}_R(\langle Q \rangle)]_0 = [H_0, \tilde{\chi}_R(\langle Q \rangle)]_0 = -\chi'_R(\langle Q \rangle)\langle Q \rangle^{-1}Q \cdot P + B_R,$$

with bounded B_R such that $\|B_R\| = O(R^{-2})$. Using the proofs of Lemma 3.1 and of Lemma C.5, we get, for $z \in \mathbb{C} \setminus \mathbb{R}$, the operator

$$\langle Q \rangle^{\sigma-\epsilon}[(z-H)^{-1}, \tilde{\chi}_R(\langle Q \rangle)]_0 \langle Q \rangle^\sigma = -\langle Q \rangle^{\sigma-\epsilon}(z-H)^{-1}[H, \tilde{\chi}_R(\langle Q \rangle)]_0(z-H)^{-1} \langle Q \rangle^\sigma$$

is bounded and, there exist $C > 0$ such that, for all $z \in \mathbb{C} \setminus \mathbb{R}$ and all $R \geq 1$,

$$\|\langle Q \rangle^{\sigma-\epsilon}[(z-H)^{-1}, \tilde{\chi}_R(\langle Q \rangle)]_0 \langle Q \rangle^\sigma\| \leq R^{2\sigma-1-\epsilon} \frac{C}{|\Im z|^2} \left(1 + \frac{\langle \Re z \rangle}{|\Im z|}\right)^2.$$

Using (C.5) with $k = 0$, we get the boundedness of $\langle Q \rangle^{\sigma-\epsilon}[\theta(H), \tilde{\chi}_R(\langle Q \rangle)]_0 \langle Q \rangle^\sigma$ and the desired upper bound on its norm. Similarly, we can treat the operator $\langle Q \rangle^{\sigma-\epsilon}[P\theta(H), \tilde{\chi}_R(\langle Q \rangle)]_0 \langle Q \rangle^\sigma$. \square

APPENDIX D. STRONGLY OSCILLATING TERM.

In this section, we focus on the case $\alpha > 1$ and prove the key result on oscillations, namely Proposition 2.4. To this end, we recall the following well-known result.

Lemma D.1. *Schur's lemma.*

Let $(n; m) \in (\mathbb{N}^*)^2$. Let $K : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{C}$ be a measurable function such that, there exists $C > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^m} |K(x; y)| dy \leq C \quad \text{and} \quad \sup_{y \in \mathbb{R}^m} \int_{\mathbb{R}^n} |K(x; y)| dx \leq C.$$

Then the operator $A : L^2(\mathbb{R}^m) \longrightarrow L^2(\mathbb{R}^n)$, that maps $f \in L^2(\mathbb{R}^m)$ to the function

$$x \mapsto \int_{\mathbb{R}^m} K(x; y) \cdot f(y) dy,$$

is well-defined, bounded and its operator norm is bounded above by C .

Proof of Proposition 2.4. Recall that, by (2.1), denoting $1 - \kappa$ by χ ,

$$e_\pm^\alpha(Q) = \left(1 - \kappa(|Q|)\right) e^{\pm i k |Q|^\alpha} = \chi(|Q|) e^{\pm i k |Q|^\alpha},$$

where $\kappa \in C_c^\infty(\mathbb{R}; \mathbb{R})$ is identically 1 near 0. Note that, for $\epsilon, \delta > 0$, $\langle Q \rangle^{-\epsilon} \langle P \rangle^{-\delta}$ is compact on $L^2(\mathbb{R}^d; \mathbb{C})$. By pseudodifferential calculus (or commutator expansions, cf. [GJ1]), $\langle Q \rangle^{-\epsilon} \langle P \rangle^{-\ell} \langle Q \rangle^\epsilon$ is bounded on $L^2(\mathbb{R}^d; \mathbb{C})$ for any $\ell \geq 0$. Thus, the desired result follows from the boundedness on $L^2(\mathbb{R}^d; \mathbb{C})$ for all $p \geq 0$ of $\langle P \rangle^{-\ell_1} \langle Q \rangle^p e_\pm^\alpha(Q) \langle P \rangle^{-\ell_2}$, for appropriate ℓ_1 and ℓ_2 . Given p , we seek for $\ell_1, \ell_2 \geq 0$ and $C > 0$ such that, for all function $f \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$, the Schwarz space on \mathbb{R}^d ,

$$\|\langle P \rangle^{-\ell_1} \langle Q \rangle^p e_\pm^\alpha(Q) \langle P \rangle^{-\ell_2} f\|^2 = \langle \langle P \rangle^{-\ell_2} f, \langle Q \rangle^p e_\mp^\alpha(Q) \langle P \rangle^{-2\ell_1} \langle Q \rangle^p e_\pm^\alpha(Q) \langle P \rangle^{-\ell_2} f \rangle$$

is bounded above by $C\|f\|^2$.

Given $f \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$, we set $g = \langle P \rangle^{-\ell_2} f \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$ and write

$$(D.1) \quad \begin{aligned} f_1(x) &:= (\langle Q \rangle^p e_{\mp}^{\alpha}(Q) \langle P \rangle^{-2\ell_1} \langle Q \rangle^p e_{\pm}^{\alpha}(Q) g)(x) \\ &= (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i\varphi_{\alpha, \pm}(x; y; \xi)} \langle x \rangle^p \chi(x) \langle \xi \rangle^{-2\ell_1} \langle y \rangle^p \chi(y) g(y) dy d\xi, \end{aligned}$$

where $\varphi_{\alpha, \pm}(x; y; \xi) = (x-y) \cdot \xi \mp k(|x|^{\alpha} - |y|^{\alpha})$ and the integral converges absolutely, if $\ell_1 > d/2$. Take $\delta \in]0; 1/2[$ and $\tau \in \mathcal{C}_c^{\infty}(\mathbb{R})$ such that $\tau(t) = 1$ if $|t| \leq 1 - 2\delta$ and $\tau(t) = 0$ if $|t| \geq 1 - \delta$. On the support of $(x; y) \mapsto \chi(x)\chi(y)\tau(|x-y| \cdot |x|^{-1})$, $|x-y| \leq (1-\delta)|x|$. In particular, on this support, 0 does not belong the segment $[x; y]$ and, for all $t \in [0; 1]$,

$$(D.2) \quad (2-\delta)|x| \geq |tx + (1-t)y| \geq |x| - (1-t)|y-x| \geq \delta|x|.$$

We write $f_1(x) = f_2(x) + f_3(x)$ where f_2 (resp. f_3) is given by (D.1) with $g(y)$ replaced by $(1 - \tau(|x-y| \cdot |x|^{-1}))g(y)$ (resp. $\tau(|x-y| \cdot |x|^{-1})g(y)$). On the support of the function $(x; y) \mapsto \chi(x)\chi(y)(1 - \tau(|x-y| \cdot |x|^{-1}))$, $|x-y| \geq (1-2\delta)|x| > 0$ and $|x-y| \geq C_{\delta}|y|$, for some $(x; y)$ -independent, positive constant C_{δ} . Since

$$(L_{x, y, D_{\xi}} - 1)e^{i(x-y) \cdot \xi \mp ik(|x|^{\alpha} - |y|^{\alpha})} = 0 \quad \text{for} \quad L_{x, y, D_{\xi}} = |x-y|^{-2}(x-y) \cdot D_{\xi},$$

we get, by integration by parts, that, for all $n \in \mathbb{N}$,

$$\begin{aligned} f_2(x) &= (2\pi)^{-d} \int e^{i\varphi_{\alpha, \pm}(x; y; \xi)} \langle x \rangle^p \chi(x) \langle y \rangle^p \chi(y) g(y) (1 - \tau(|x-y| \cdot |x|^{-1})) \\ &\quad \cdot (L_{x, y, D_{\xi}}^*)^n (\langle \xi \rangle^{-2\ell_1}) dy d\xi. \end{aligned}$$

Choosing n large enough, we can apply Lemma D.1 to show that the map $f \mapsto f_2$ is bounded on $L^2(\mathbb{R}^d)$.

On the support of the function $(x; y) \mapsto \chi(x)\chi(y)\tau(|x-y| \cdot |x|^{-1})$, we can write $\varphi_{\alpha, \pm}(x; y; \xi) = (x-y) \cdot (\xi \mp kw_{\alpha}(x; y))$ where

$$w_{\alpha}(x; y) = \alpha \int_0^1 |tx + (1-t)y|^{\alpha-2} (tx + (1-t)y) dt.$$

Setting, for $j \in \{0; 1\}$,

$$\lambda_j = \int_0^1 |tx + (1-t)y|^{\alpha-2} t^j dt,$$

$\lambda_0 \geq \lambda_1 > 0$ and $\alpha^{-1}w_{\alpha}(x; y) = \lambda_1 x + (\lambda_0 - \lambda_1)y = \lambda_0((\lambda_1/\lambda_0)x + (1 - \lambda_1/\lambda_0)y)$. By (D.2),

$$\lambda_0 \geq \lambda_1 \geq 2^{-1}(\delta|x|)^{\alpha-2}$$

and $|w_{\alpha}(x; y)| \geq \alpha\lambda_0\delta|x|$. Furthermore $|w_{\alpha}(x; y)| \leq \alpha((2-\delta)|x|)^{\alpha-1}$, thus

$$(D.3) \quad 2^{-1}\delta^{\alpha-1} \leq \alpha^{-1}|x|^{1-\alpha} \cdot |w_{\alpha}(x; y)| \leq (2-\delta)^{\alpha-1},$$

$$(D.4) \quad 2^{-1}\delta^{\alpha-1}(2-\delta)^{1-\alpha} \leq \alpha^{-1}|y|^{1-\alpha} \cdot |w_{\alpha}(x; y)| \leq \delta^{1-\alpha}(2-\delta)^{\alpha-1}.$$

In the integral defining f_3 , we make the change of variables $\xi \mapsto \eta = \xi \mp kw_{\alpha}(x; y)$ and obtain

$$(D.5) \quad \begin{aligned} f_3(x) &= (2\pi)^{-d} \int e^{i(x-y) \cdot \eta} \langle x \rangle^p \chi(x) \langle y \rangle^p \chi(y) g(y) \tau(|x-y| \cdot |x|^{-1}) \\ &\quad \cdot \langle \eta \pm kw_{\alpha}(x; y) \rangle^{-2\ell_1} dy d\eta. \end{aligned}$$

We write $f_3(x) = f_4(x) + f_5(x)$ where f_4 (resp. f_5) is given by (D.5) with $g(y)$ replaced by $\tau(|\eta| \cdot |kw_\alpha(x; y)|^{-1})g(y)$ (resp. $(1 - \tau(|\eta| \cdot |kw_\alpha(x; y)|^{-1}))g(y)$). On the support of the integrand of f_4 , $|\eta| \leq (1 - \delta)|kw_\alpha(x; y)|$ which implies that $|\eta \pm kw_\alpha(x; y)| \geq \delta|kw_\alpha(x; y)|$. Take $\ell_1 > (\alpha - 1)^{-1}(p + d)$. By (D.3), (D.4), and Lemma D.1, the map $f \mapsto f_4$ is bounded on $L^2(\mathbb{R}^d)$.

On the support of the integrand of f_5 , $|\eta| \geq (1 - 2\delta)|kw_\alpha(x; y)| > 0$. Since

$$M_{\eta, D_x} e^{i(x-y) \cdot \eta} = e^{i(x-y) \cdot \eta} = -M_{\eta, D_y} e^{i(x-y) \cdot \eta} \quad \text{for} \quad M_{\eta, D_z} = |\eta|^{-2} \eta \cdot D_z,$$

we get, by integration by parts, that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \langle g, f_4 \rangle &= (2\pi)^{-d} \int e^{i(x-y) \cdot \eta} \left(-M_{\eta, D_x}^* M_{\eta, D_y}^* \right)^n \left[\langle x \rangle^p \chi(x) \overline{g(x)} \langle y \rangle^p \chi(y) g(y) \right. \\ &\quad \left. \cdot \tau(|x - y| \cdot |x|^{-1}) (1 - \tau(|\eta| \cdot |kw_\alpha(x; y)|^{-1})) \right] dx dy d\eta. \end{aligned}$$

Choosing the integer n such that $n(\alpha - 1) > p + d$, using (D.3) and (D.4), we can apply Lemma D.1 to get some f -independent constant $C_0 > 0$ such that

$$|\langle g, f_4 \rangle| \leq C_0 \sup_{0 \leq |\gamma| \leq n} (\|g\|^2 + \|P^\gamma g\|^2).$$

Now the r.h.s. is bounded above by $C\|f\|^2$ if ℓ_2 is greater than 1 plus the integer part of $(\alpha - 1)^{-1}(p + d)$. \square

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